Proof mining in CAT(0)-spaces

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Talk

• case study in the general project of *proof mining* in functional analysis

• quadratic rate of asymptotic regularity for the Krasnoselski-Mann iterations of nonexpansive mappings in CAT(0)-spaces

Hyperbolic spaces

[Takahashi, Goebel/Kirk, Reich/Shafrir, Kohlenbach]

A **hyperbolic space** is a triple (X, ρ, W) where (X, ρ) is metric space and $W : X \times X \times [0,1] \to X$ s.t.

$$\begin{array}{ll} (W1) & \rho(z, W(x,y,\lambda)) \le (1-\lambda)\rho(z,x) + \lambda\rho(z,y), \\ (W2) & \rho(W(x,y,\lambda), W(x,y,\tilde{\lambda})) = |\lambda - \tilde{\lambda}| \cdot \rho(x,y), \\ (W3) & W(x,y,\lambda) = W(y,x,1-\lambda), \\ (W4) & \rho(W(x,z,\lambda), W(y,w,\lambda)) \le (1-\lambda)\rho(x,y) + \lambda\rho(z,w). \end{array}$$

Notation: $(1-\lambda)x \oplus \lambda y := W(x,y,\lambda)$.
$[x,y] := \{W(x,y,\lambda) : \lambda \in [0,1]\}$.

CAT(0)-spaces

- defined by Gromov in 1981
- nice geometric properties; arrise in a wide variety of contexts
- CAT(0)-space = hyperbolic space (X, ρ, W) satisfying the

so-called CN-inequality of Bruhat-Tits: for all $x, y, z \in X$,

$$\rho\left(z, \frac{1}{2}x \oplus \frac{1}{2}y\right)^2 \leq \frac{1}{2}\rho(z,x)^2 + \frac{1}{2}\rho(z,y)^2 - \frac{1}{4}\rho(x,y)^2.$$

- an analog to Hilbert spaces

Asymptotic regularity

(X, ρ, W) hyperbolic, $C \subseteq X$ convex, $T : C \to C$ nonexpansive, (x_n) the Krasnoselski-Mann iteration starting from $x \in C$:

$$x_0 := x, \quad x_{n+1} := (1 - \lambda_n)x_n \oplus \lambda_n T x_n$$

where $(\lambda_n)_n$ sequence in $[0, 1]$.

• asymptotic regularity - defined by Browder/Petryshyn(66) for normed spaces

• $T : C \to C$ is *asymptotically regular* if for all $x \in C$,

$$\lim_{n\to\infty} \rho(T^n(x), T^{n+1}(x)) = 0.$$

- for constant $\lambda_n = \lambda \in [0,1]$

$$\lim_{n\to\infty} \rho(x_n, Tx_n) = 0 \text{ for all } x \in C$$

$$\Updownarrow$$

asymptotic regularity of the averaged mapping

$$T_\lambda := (1-\lambda)I \oplus \lambda T.$$

- for general $\lambda_n \in [0,1]$, T is λ_n-*asymptotically regular* if for all $x \in C$,

$$\lim_{n\to\infty} \rho(x_n, Tx_n) = 0.$$

- asymptotic regularity for normed spaces and bounded C:
  - constant $\lambda_n = \lambda$: Edelstein/O'Brien(78)
  - general λ_n: Ishikawa(76), Goebel/Kirk(83)

Rates of asymptotic regularity

- for normed spaces and $\lambda_n = \lambda \in (0,1)$:
  - uniform and optimal quadratic bound - Baillon/Bruck(96)
  - computer aided proof
  - classical proof only for $\lambda_n = 1/2$ - Bruck(96)
- **open problem** in Baillon/Bruck(96):

  do these methods hold for general λ_n or for hyperbolic spaces?
- exponential bound for general λ_n:
  - normed spaces - Kohlenbach (01)
  - hyperbolic spaces - Kohlenbach/L.(03)

Groetsch's theorem

Groetsch(72)

$(X, \|\cdot\|)$ uniformly convex normed space, $C \subseteq X$ nonempty convex, $T : C \to C$ nonexpansive s. t. T has at least one fixed point.
$(\lambda_n) \subseteq [0,1]$ satisfies the following condition

$$\sum_{k=0}^{\infty} \lambda_k(1-\lambda_k) = \infty. \qquad (1)$$

Then T is λ_n-asymptotically regular.

- (1) $\Leftrightarrow$ there exists a witness $\theta : \mathbb{N} \to \mathbb{N}$ s. t. for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{\theta(n)} \lambda_k(1-\lambda_k) \geq n.$$

Groetsch's theorem - proof mining

- Kohlenbach(03) - quantitative version of a strengthening of Groetsch's theorem: assume only the existence of approximate fixed points in some neighborhood of x

- extend Kohlenbach's results to *uniformly convex* hyperbolic spaces

- specialize to CAT(0)-spaces

Uniformly convex hyperbolic spaces

(X, ρ, W) is *uniformly convex* if for any $r > 0$, and $\varepsilon \in (0, 2]$ there exists a $\delta \in (0, 1]$ s. t. for all $a, x, y \in X$,

$$\left.\begin{array}{l} \rho(x,a) \leq r \\ \rho(y,a) \leq r \\ \rho(x,y) \geq \varepsilon r \end{array}\right\} \quad \Rightarrow \quad \rho\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq (1-\delta)r.$$

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta := \eta(r, \varepsilon)$ is called a *modulus of uniform convexity*.

Uniformly convex hyperbolic spaces

(X, ρ, W) uniformly convex with modulus of uniform convexity η.

For any $r > 0, \varepsilon \in (0,2], \lambda \in [0,1]$, and $a, x, y \in X$,

$$\left.\begin{array}{l} \rho(x,a) \leq r \\ \rho(y,a) \leq r \\ \rho(x,y) \geq \varepsilon r \end{array}\right\} \quad \Rightarrow \quad \rho((1-\lambda)x \oplus \lambda y, a) \leq (1 - 2\lambda(1-\lambda)\eta(r,\varepsilon))r.$$

Main Theorem

(X, ρ, W) uniformly convex hyperbolic space with modulus η,
η decreases with r.
$C \subseteq X$ nonempty convex subset, $T : C \to C$ nonexpansive.
$(\lambda_n) \subseteq [0, 1]$, $\theta : \mathbb{N} \to \mathbb{N}$ s.t. for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{\theta(n)} \lambda_k(1 - \lambda_k) \geq n. \tag{2}$$

$x \in C, b > 0$ s. t. for any $\delta > 0$ there is $y \in C$ with

$$\rho(x, y) \leq b, \quad \text{and} \quad \rho(y, Ty) < \delta. \tag{3}$$

Then

$$\lim_{n\to\infty} \rho(x_n, Tx_n) = 0,$$

and moreover

$$\forall \varepsilon > 0 \, \forall n \geq h(\varepsilon, \eta, b, \theta) \left(\rho(x_n, Tx_n) \leq \varepsilon\right), \tag{4}$$

where

$$h(\varepsilon, \eta, b, \theta) := \begin{cases} \theta\left(\left\lceil \dfrac{b+1}{\varepsilon \cdot \eta\left(b+1, \frac{\varepsilon}{b+1}\right)} \right\rceil\right) & \text{for } \ \varepsilon < 2b \\ 0 & \text{otherwise.} \end{cases}$$

Main theorem

Assume moreover that

$$\eta(r,\varepsilon) = \varepsilon \cdot \tilde{\eta}(r,\varepsilon)$$

s.t. $\tilde{\eta}$ increases with ε.

Then the bound $h(\varepsilon,\eta,b,\theta)$ can be replaced for $\varepsilon < 2b$ by

$$\tilde{h}(\varepsilon,\eta,b,\theta) := \theta\left(\left\lceil \frac{b+1}{2\varepsilon \cdot \tilde{\eta}\left(b+1, \frac{\varepsilon}{b+1}\right)} \right\rceil\right).$$

Groetsch's theorem for UC hyperbolic spaces

Consequence

(X,ρ,W) uniformly convex hyperbolic space, $C \subseteq X$ nonempty convex subset

$T : C \to C$ nonexpansive s. t. T has at least one fixed point

$(\lambda_n) \subseteq [0,1]$ s.t.

$$\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n) = \infty.$$

Then for any $x \in C$,

$$\lim_{n\to\infty} \rho(x_n, Tx_n) = 0.$$

Bounded C

Consequence

$(X, \rho, W), C, T, (\lambda_n), \theta$ are as in the hypothesis of Main Theorem.
C bounded with finite diameter d_C
Then T is λ_n-asymptotically regular, and for all $x \in C$,

$$\forall \varepsilon > 0 \, \forall n \geq h(\varepsilon, \eta, d_C, \theta) \left(\rho(x_n, Tx_n) \leq \varepsilon \right),$$

where $h(\varepsilon, \eta, d_C, \theta)$ is as above with d_C instead of b.

$\Rightarrow$ uniform rate of asymptotic regularity for general (λ_n)

CAT(0)-spaces

Let X be a CAT(0)-space, $C \subseteq X$ convex **bounded** with diameter d_C, and $T : C \to C$ nonexpansive.

- X is uniformly convex with modulus of uniform convexity:

$$\eta(r, \varepsilon) := \frac{\varepsilon^2}{8}$$

- η decreases with r
- $\eta(r, \varepsilon) = \varepsilon \cdot \tilde{\eta}(r, \varepsilon)$, and $\tilde{\eta}$ increases with ε

CAT(0)-spaces-general λ_n

Consequence

$(\lambda_n), \theta$ are as in the hypothesis of Main Theorem.
Then T is λ_n-asymptotically regular, and for all $x \in C$,

$$\forall \varepsilon > 0 \, \forall n \geq g(\varepsilon, d_C, \theta) \left(\rho(x_n, Tx_n) \leq \varepsilon\right), \tag{5}$$

where

$$g(\varepsilon, d_C, \theta) := \begin{cases} \theta\left(\left\lceil \dfrac{8(d_C+1)^2}{\varepsilon^2} \right\rceil\right) & \text{for } \ \varepsilon < 2d_C \\ 0 & \text{otherwise.} \end{cases}$$

CAT(0)-spaces-constant λ_n

Consequence

$(\lambda_n) = \lambda \in (0,1)$.
Then T is λ-asymptotically regular, and for all $x \in C$,

$$\forall \varepsilon > 0\, \forall n \geq \tilde{g}(\varepsilon, d_C, \lambda) \left(\rho(x_n, Tx_n) \leq \varepsilon\right), \tag{6}$$

where

$$\tilde{g}(\varepsilon, d_C, \lambda) := \begin{cases} \dfrac{1}{\lambda(1-\lambda)} \left\lceil \dfrac{8(d_C+1)^2}{\varepsilon^2} \right\rceil & \text{for } \ \varepsilon < 2d_C \\ 0 & \text{otherwise.} \end{cases}$$

quadratic (in $1/\varepsilon$) rate of asymptotic regularity

Effective bounds for asymptotic regularity

	$\lambda_n = \lambda$	general λ_n
Hilbert	quadratic: Browder/Petryshyn(67)	$\theta\left(\frac{1}{\varepsilon^2}\right)$: K.(03)
UC normed	K.(03), Kirk/Martinez(90)	K.(03)
normed	quadratic: Baillon/Bruck(96)	K.(01)
CAT(0)	**quadratic**: L.(05)	$\theta\left(\boldsymbol{\frac{1}{\varepsilon^2}}\right)$: L.(05)
UC hyperbolic	L.(05)	L.(05)
hyperbolic	exponential: K./L.(03)	K./L.(03)