Operadic Algebraic Topology

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1 Introduction

The main method of algebraic topology is to assign to a topological space certain algebraic object (model) and to study this relatively simple algebraic object instead of complex geometric one.

Examples of such models are chain and cochain complexes, homology and homotopy groups, cohomology algebra, etc.

The main problem here is to find models that classify spaces up to some equivalence relation, such as homeomorphism, homotopy equivalence, rational homotopy equivalence (an equivalence relation generate by maps that induce isomorphisms of rational homology), etc.

Usually such models are not *complete*: the equivalence of models does not guarantee the equivalence of spaces. They can just distinguish spaces.

The models which carry richer algebraic structure contain more information about the space. For example the model "cohomology algebra" allows to distinguish spaces, which can not be distinguished by the model "cohomology groups".

One can not expect the existence of more or less simply complete algebraic models in general case but for the rational homotopy equivalence there are various complete homotopy invariants due to Quillen and Sullivan.

The key point here is the existence in the rational case of *commutative cochains*. Two 1-connected spaces are rationally homotopy equivalent if and only if their commutative cochain algebras are weak equivalent .

But outside of rational case the situation is much more complicated. The ordinary (noncommutative) cochain complex is to poor to determine homo-

[1]ICTP Map Summer School, 11-29 August 2008, Trieste

topy type. The structure of differential graded algebra must be enriched with new cochain operations, such as Steenrod operations, which measure the deviation from commutativity. But this structure also is not enough. The further enrichment requires huge structure of cochain operation. The operadic technics is appropriate tool to handle such huge structures. The final result in this direction is the result of Mandell stating that for some class of topological spaces cochain complex equipped by a structure of algebra over so called E_∞-operad determines homotopy type.

The organization is as follows.

In Section2 the notions of chain and cochain complexes are presented. In Section 3 the differential algebras and coalgebras are defined. In Section 4 the bar and cobar constructions are introduced. In Section 5 A_∞-algebras are discussed. In section 6 Steenrod cochain operations are presented. In Section 7 homotopy G-algebras are discussed and finally Section 8 is dedicated to differential graded operads.

2 Chain and Cochain Complexes

2.1 Graded Modules

We work over commutative associative ring with unit R.

A *graded module* is a collection of R-modules

$$..., M_{-1}, M_0, M_1, ..., M_n, M_{n+1}, ... \quad .$$

A *morphism of graded modules* $M_* \to M'_*$ is a collection of homomorphisms $\{f_i : M_i \to M'_i,\ i \in Z\}$.

Sometimes we use the following notion: a *morphism of graded modules of degree* n is a collection of homomorphisms $\{f_i : M_i \to M'_{i+n},\ i \in Z\}$. So a morphism of graded modules has the degree 0.

2.2 Chain Complexes

Definition 1 *A differential graded (dg) module (or a chain complex) is a sequence of R modules and homomorphisms*

$$... \overset{d_{-1}}{\leftarrow} C_{-1} \overset{d_0}{\leftarrow} C_0 \overset{d_1}{\leftarrow} C_1 \overset{d_2}{\leftarrow} ... \overset{d_{n-1}}{\leftarrow} C_{n-1} \overset{d_n}{\leftarrow} C_n \overset{d_{n+1}}{\leftarrow} C_{n+1} \overset{d_{n+2}}{\leftarrow} ... \quad .$$

such that $d_i d_{i+1} = 0$.

Elements of C_n are called *n-dimensional chains*; homomorphisms d_i are called *boundary operators*, or *differentials*; elements of $Z_n = Ker\ d_n \subset C_n$ are called *n-dimensional cycles* and elements of $B_n = Im\ d_{n+1} \subset C_n$ are called *n-dimensional boundaries*.

It follows from the condition $d_i d_{i+1} = 0$ that $B_n \subset Z_n$.

Definition 2 *The* n-th homology module $H_n(C_*)$ *of a dg module* (C_*, d_*) *is defined as the factor* Z_n/B_n.

A sequence $C_{n-1} \stackrel{d_n}{\leftarrow} C_n \stackrel{d_{n+1}}{\leftarrow} C_{n+1}$ is *exact*, that is $B_n = Z_n$, iff $H_n(C_*) = 0$. Thus homology measures the deviation from the exactness.

2.3 Cochain Complexes

The notion of *cochain complex* differs from the notion of chain complex by direction of the differential

$$... \stackrel{d^{-1}}{\to} C^{-1} \stackrel{d^0}{\to} C^0 \stackrel{d^1}{\to} C^1 \stackrel{d^2}{\to} ... \stackrel{d^{n-1}}{\to} C^{n-1} \stackrel{d^n}{\to} C^n \stackrel{d^{n+1}}{\to} C^{n+1} \stackrel{d^{n+2}}{\to} ... \quad .$$

Corresponding terms here are cochains, cocycles, coboundaries, cohomology.

Changing indices $C^n = C_{-n}$, $d^n = d_{-n}$ we convert a chain complex (C_*, d_*) to a cochain complex (C^*, d^*).

2.4 Dual Cochain Complex

For a chain complex (C_*, d_*) and an R-module A the *dual cochain complex* $C^* = (Hom(C_*, A), \delta^*)$ is defined as

$$C^n = Hom(C_n, A), \quad \delta^*(\phi) = \phi d.$$

2.5 Chain maps

Definition 3 *A* chain map *of chain complexes* $f : (C_*, d_*) \to (C'_*, d'_*)$ *is defined as a sequence of homomorphisms* $\{f_i : C_i \to C'_i\}$ *such that* $d'_n f_n = f_{n-1} d_n$.

This condition means the commutativity of the diagram

$$\begin{array}{ccccccccc} \dots & \overset{d_{n-1}}{\leftarrow} & C_{n-1} & \overset{d_n}{\leftarrow} & C_n & \overset{d_{n+1}}{\leftarrow} & C_{n+1} & \overset{d_{n+2}}{\leftarrow} & \dots \\ & & \downarrow f_{n-1} & & \downarrow f_n & & \downarrow f_{n+1} & & \\ \dots & \overset{d'_{n-1}}{\leftarrow} & C'_{n-1} & \overset{d'_n}{\leftarrow} & C'_n & \overset{d'_{n+1}}{\leftarrow} & C'_{n+1} & \overset{d'_{n+2}}{\leftarrow} & \dots . \end{array}$$

Proposition 1 *The composition of chain maps is a chain map.*

Chain complexes and chain maps form a category which we denote by $DGMod$.

Proposition 2 *If* $\{f_i\} : (C_*, d_*) \to (C'_*, d'_*)$ *is a chain map, then* f_n *sends cycles to cycles and boundaries to boundaries, i.e.* $f_n(Z_n) \subset Z'_n$ *and* $f_n(B_n) \subset B'_n$*.*

Proposition 3 *A chain map* $\{f_i\} : (C_*, d_*) \to (C'_*, d'_*)$ *induces the correct homomorphism of homology groups*

$$f_n^* : H_n(C_*) \to H_n(C'_*).$$

Homology is a functor from the category of dg modules to the category of graded modules

$$H : DGMod \to GMod.$$

2.6 *Hom* Complex

For two chain complexes $C,\ C'$ define the chain complex $(Hom(C, C'), D)$ as

$$Hom(C, C')_n = Hom^n(C, C')$$

where $Hom^m(C, C') = \{\phi : C_* \to C'_{*+n},\}$ is the module of homomorphisms of degree n, and the differential $D : Hom^n(C, C') \to Hom^{n-1}(C, C')$ is given by $D(\phi) = d'\phi + (-1)^{deg\ \phi}\phi d$.

2.7 Tensor Product

For two chain complexes A and B the tensor product $A \otimes B$ is defined as the following chain complex:

$$(A \otimes B)_n = \sum_{p+q=n} A_p \otimes B_q,$$

with differential $d_\otimes : (A \otimes B)_n \to (A \otimes B)_{n-1}$ given by

$$d_\otimes(a_p \otimes b_q) = d_p(a_p) \otimes b_q + (-1)^p a_p \otimes d'_q(b_q).$$

If $f : A \to A'$ and $g : B \to B'$ are chain maps then there is a chain map

$$f \otimes g : A \otimes B \to A' \otimes B'$$

defined as $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$.

2.8 Chain Homotopy

Definition 4 *Chain maps* $\{f_i\}, \{g_i\} : (C_*, d_*) \to (C'_*, d'_*)$ *are called* chain homotopic*, if there exists a sequence of homomorphisms* $D_n : C_n \to C'_{n+1}$*,*

$$\begin{array}{ccccccccc}
\ldots & \overset{d_{n-1}}{\longleftarrow} & C_{n-1} & \overset{d_n}{\longleftarrow} & C_n & \overset{d_{n+1}}{\longleftarrow} & C_{n+1} & \overset{d_{n+2}}{\longleftarrow} & \ldots \\
 & & f_{n-1} \downarrow\downarrow g_{n-1} & \searrow D_{n-1} & f_n \downarrow\downarrow g_n & \searrow D_n & f_{n+1} \downarrow\downarrow g_{n+1} & & \\
\ldots & \overset{d'_{n-1}}{\longleftarrow} & C'_{n-1} & \overset{d'_n}{\longleftarrow} & C'_n & \overset{d'_{n+1}}{\longleftarrow} & C'_{n+1} & \overset{d'_{n+2}}{\longleftarrow} & \ldots .
\end{array}$$

such that $f_n - g_n = d'_{n+1} D_n + D_{n-1} d_n$*. In this case we write* $f \sim_D g$*.*

Proposition 4 *Chain homotopy is an equivalence relation:*

$$\begin{array}{ll}
(a) & f \sim_0 f; \\
(b) & f \sim_D g \Longrightarrow g \sim_{-D} f; \\
(c) & f \sim_D g,\ g \sim_{D'} h \Longrightarrow f \sim_{D+D'} h.
\end{array}$$

Proposition 5 *Chain homotopy is compatible with compositions:*

$$\begin{array}{ll}
(a) & f \sim_D g \Longrightarrow hf \sim_{hD} hg; \\
(b) & f \sim_D g \Longrightarrow fk \sim_{Dk} gk.
\end{array}$$

Thus there is the category $hoDGMod$ whose objects are chain complexes and morphisms are chain homotopy classes

$$Hom_{hoDGMod}(C, C') = [C, C'] = Hom_{DGMod}(C, C')/\sim .$$

Proposition 6 *If two chain maps* $\{f_i\}, \{g_i\} : (C_*, d_*) \to (C'_*, d'_*)$ *are chain homotopic, then the induced homomorphism of homology groups coincide:* $f_n^* = g_n^* : H_n(C_*) \to H_n(C'_*)$.

Thus we have the commutative diagram of functors

$$\begin{array}{ccc} DGMod & \longrightarrow & hoDGMod \\ H \searrow & & \swarrow H \\ & GMod. & \end{array}$$

2.9 Chain Equivalence

Chain complexes C and C' are called *chain equivalent* $C \sim C'$, if there exist chain maps

$$f : C \leftrightarrows C' : g$$

such that $gf \sim id_C$, $fg \sim id_{C'}$. This means that C and C' are isomorphic in $hoDGMod$.

A chain complex C is called *contractible* if $C \sim 0$, equivalently if $id_C \sim 0 : C \to C$.

Proposition 7 *Each contractible* C *is acyclic, i.e.* $H_i(C) = 0$ *for all* i*-s.*

Proposition 8 *If all* C_i*-s are free and* C *is acyclic, then* C *is contractible.*

2.10 Algebraic Example

Let $(A, \mu : A \otimes A \to A)$ be an associative algebra, then

$$C(A) = (A \overset{\mu}{\leftarrow} A \otimes A \overset{\mu\otimes id - id\otimes\mu}{\leftarrow} A \otimes A \otimes A \overset{\mu\otimes id\otimes id - id\otimes\mu\otimes id + id\otimes id\otimes\mu}{\leftarrow} ...)$$

is a chain complex: the associativity condition guarantees that $dd = 0$.

If A has a unit $e \in A$ then this complex is contractible, that is $id : C(A) \to C(A)$ and $0 : C(A) \to C(A)$ are homotopic: the suitable chain homotopy is given by $D(a_1 \otimes ... \otimes a_n) = (e \otimes a_1 \otimes ... \otimes a_n)$. This immediately implies that $C(A)$ is acyclic, that is $H_i(C(A)) = 0$ for all $i > 0$.

This example is a particular case of more general chain complex called bar construction, see lather.

2.11 Topological Example

2.11.1 Simplicial Complexes

Simplicial complex is a formal construction, which models topological spaces.

Definition 5 *A* simplicial complex *is a set V with a given family of finite subsets, called simplexes, so that the following conditions are satisfied:*
(1) all points of V are simplexes;
(2) any nonempty subset of a simplex is a simplex.

A simplex consisting of $(n+1)$ points is called *n-dimensional simplex*. The 0-dimensional simplexes, i.e. the points of V are called *vertexes*.

Definition 6 *A* simplicial map *of simplicial complexes $V \to V'$ is a map of vertexes $f : V \to V'$ such that the image of any simplex of V is a simplex in V'.*

Proposition 9 *The composition of simplicial maps is a simplicial map.*

Simplicial complexes and simplicial maps form a category which we denote as SC.

Proposition 10 *To any simplicial set V corresponds a topological space $|V|$ (called it's* realization*) and to any simplicial map $f : V \to V'$ corresponds a continuous map of realizations $|f| : |V| \to |V'|$.*

So the realization is a functor from the category of simplicial complexes to the category of topological spaces

$$| - | : SC \to Top.$$

2.11.2 Homology Modules of a Simplicial Complex

In this section we consider *ordered* simplicial complexes: we assume that the set of vertexes V is ordered by any order.

We assign to a such ordered simplicial complex the following chain complex $(C_*(V), d_*)$: Let $C_n(V)$ be the free R-module, generated by all ordered

n-simplexes $\sigma_n = (v_{k_0}, v_{k_1}, ..., v_{k_n})$, where $v_{k_0} < v_{k_1} < ... < v_{k_n}$; the differential $d_n : C_n(V) \to C_{n-1}(V)$ on a generator $\sigma_n = (v_{k_0}, v_{k_1}, ..., v_{k_n}) \in C_n(V)$ is given by

$$d_n(v_{k_0}, v_{k_1}, ..., v_{k_n}) = \sum_{i=0}^{n} (-1)^i (v_{k_0}, ..., \widehat{v}_{k_i}, ..., v_{k_n}),$$

where $(v_{k_0}, ..., \widehat{v}_{k_i}, ..., v_{k_n})$ is the $(n-1)$-simplex, obtained by omitting of v_{k_i}, and is extended on the whole $C_n(V)$ linearly.

Proposition 11 *The composition $d_{n-1}d_n$ is a zero map, thus $(C_*(V), d_*)$ is a chain complex.*

Definition 7 *The n-*th homology group *$H_n(V)$ of an ordered simplicial set V is defined as the n-th homology group $H_n(C_*(V))$.*

2.11.3 Cohomology Modules of a Simplicial Complex

Let A be an R-module. The cochain complex of V with coefficients in A is defined as dual to the chain complex $C_*(V)$: $C^*(V, A) = Hom(C_*(V), A)$. The n-th cohomology module of V with coefficients in A is just the n-th homology of this cochain complex.

Bellow we show that cohomology $H^*(V, A)$ is more interesting then homology $H_*(V)$, since cohomology possesses richer algebraic structure: it is a ring.

3 Differential Graded Algebras and Coalgebras

3.1 Graded Algebras

A *graded algebra* is a graded module

$$..., A_{-1}, A_0, A_1, ..., A_n, A_{n+1}, ...$$

equipped with associative multiplication

$$\mu : A_p \otimes A_q \to A_{p+q}.$$

We denote $a \cdot b = \mu(a \otimes b)$.

For a graded algebra $\{A_i\}$ the component A_0 is an associative algebra.

A *morphism of graded algebras* $f : A \to A'$ is a morphism of graded modules $\{f_i : A_i \to A'_i,\ i \in Z\}$ which is *multiplicative*, that is $f(a \cdot b) = f(a) \cdot f(b)$.

If $f, g : A \to A'$ are two morphisms of graded algebras, then an (f, g)-*derivation of degree* k is defined as a morphism of degree k $D : A_* \to A_{*+k}$ (i.e. a collection of homomorphisms $\{D_i : A_i \to A_{i+k},\ i \in Z\}$, which satisfies the condition

$$D(a \cdot b) = D(a) \cdot g(b) + (-1)^{k \cdot |a|} f(a) \cdot D(b).$$

3.2 Differential Graded Algebras

Definition 8 *A differential graded algebra (dga in short) (A, d, μ) is a dg module (A, d) equipped additionally with a multiplication*

$$\mu : A \otimes A \to A$$

so that (A, μ) is a graded algebra, and the multiplication μ and the differential d are connected by the condition

$$d(a \cdot b) = da \cdot b + (-1)^{|a|} a \cdot db.$$

This condition means simultaneously that μ is a chain map, and that d is a (id, id)-derivation of degree 1.

A morphism of dga-s $f : (A, d, \mu) \to (A, d, \mu)$ is defined as a multiplicative chain map:

$$df = fd, \quad f(a \cdot b) = f(a) \cdot f(b).$$

For a dg algebra (A, d, μ) its homology $H(A)$ is a graded algebra with the following multiplication:

$$H_*(A) \otimes H_*(A) \xrightarrow{\phi} H_*(A \otimes A) \xrightarrow{H(\mu)} H_*(A),$$

were $\phi : H_*(A) \otimes H_*(A) \to H_*(A \otimes A)$ is the standard map

$$\phi(h_1 \otimes h_2) = cl(z_{h_1} \otimes z_{h_2}).$$

In other words the multiplication on $H(A)$ is defined as follows: For $h_1, h_2 \in H(A)$ the product $h_1 \cdot h_2$ is the homology class of the cycle $z_{h_1} \cdot z_{h_2}$.

Furthermore, a dga map induces a multiplicative map of homology graded algebras.

Thus homology is a functor from the category of dg algebras to the category of graded algebras.

3.3 Derivation Homotopy

Two dg algebra maps $f, g : A \to A'$ are called homotopic if there exists a chain homotopy $D : A \to A'$, $f - g = dD + Dd$ which, in addition is a (f, g)-derivation, that is

$$D(a \cdot b) = D(a) \cdot g(b) + (-1)^{|a|} f(a) \cdot D(b).$$

Note that generally this is not an equivalence relation.

3.4 Graded Coalgebras

A graded coalgebra (C, Δ) is a graded module

$$..., C_{-1}, C_0, C_1, ..., C_n, C_{n+1}, ...$$

equipped with a comultiplication

$$\Delta : C \otimes C \to C$$

which is coassociative, that is $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$, i.e. commutes the diagram

$$\begin{array}{cccccc} & C & \xrightarrow{\Delta} & C \otimes C & \\ \Delta & \downarrow & & \downarrow & \Delta \otimes id \\ & C \otimes C & \xrightarrow{id \otimes \Delta} & C \otimes C \otimes C & . \end{array}$$

A morphism of graded coalgebras $f : (C, \Delta) \to (C', \Delta')$ is a morphism of graded modules $\{f_k : C_k \to C'_k\}$ which is comultiplicative, that is $\Delta' f = (f \otimes f)\Delta$, i.e. commutes the diagram

$$\begin{array}{ccccc} & C & \xrightarrow{f} & C' & \\ \Delta & \downarrow & & \downarrow & \Delta' \\ & C \otimes C & \xrightarrow{f \otimes f} & C' \otimes C' & \end{array}$$

If $f, g : C \to C'$ are two morphisms of graded coalgebras, then an (f, g)-coderivation of degree k is defined as a collection of homomorphisms $\{D_i : C_i \to C_{i+k}\}$ which satisfies the condition $\Delta' D = (f \otimes D + D \otimes g)\Delta$, i.e. commutes the diagram

$$\begin{array}{ccccc} & C & \stackrel{D}{\longrightarrow} & C' & \\ \Delta & \downarrow & & \downarrow & \Delta' \\ & C \otimes C & \stackrel{f\otimes D + D\otimes g}{\longrightarrow} & C' \otimes C' & . \end{array}$$

3.5 Differential Graded Coalgebras

Definition 9 *A differential graded coalgebra (dgc) (C, d, Δ) is a dg module (C, d) equipped additionally with a comultiplication $\Delta : C \to C \otimes C$ so that (C, Δ) is a graded coalgebra and the comultiplication Δ and the differential d are connected with the condition*

$$\Delta d = (d \otimes id + id \otimes d)\Delta.$$

This condition means simultaneously that Δ is a chain map, and that d is a (id, id)-coderivation of degree 1.

A morphism of dgc-s $f : (C, d, \Delta) \to (C', d', \Delta')$ is defined as a morphism of graded coalgebras which is a chain map.

Generally for a dg coalgebra (C, d, μ) its homology $H(C)$ *is not* a graded coalgebra:

$$H_*(C) \otimes H_*(C) \stackrel{\phi}{\longrightarrow} H_*(C \otimes C) \stackrel{H(\Delta)}{\longleftarrow} H_*(C),$$

the map $\phi : H_*(C) \otimes H_*(C) \to H_*(C \otimes C)$ has wrong direction, but if all $H_i(C)$-s are free then ϕ s invertible and $H_*(C)$ is a graded coalgebra.

3.6 Duality

Let (C_*, d_*, Δ) be a dg coalgebra and let $(A, \mu_A : A \otimes A \to A)$ be an R-algebra. Consider the dual cochain complex

$$(C^* = Hom(C_*, A), \delta^*).$$

The comultiplication $\Delta : C_* \to C_* \otimes C_*$ implies the multiplication

$$\mu : Hom(C_*, A) \otimes Hom(C_*, A) \to Hom(C_*, A)$$

given by

$$\mu(\phi \otimes \psi) = \mu_A(\phi \otimes \psi)\Delta.$$

Proposition 12 *For a dg coalgebra* (C_*, d_*, Δ) *the dual cochain complex is a dg algebra* $(C^* = Hom(C_*, A)\delta^*, \mu)$.

Again this is particular case of more general construction. For a dg coalgebra (C_*, d_*, Δ_C) and for a dg algebra (A, d, μ_A) the Hom-complex $(Hom(C, A), D)$ is a dg algebra with respect to the *cup* product

$$\phi \smile \psi = \mu_A(\phi \otimes \psi)\Delta_C.$$

3.7 Alexander-Whitney Diagonal

Let V be an ordered simplicial complex and $C_*(V)$ be its chain complex. There exists a comultiplication

$$\Delta : C_*(X) \to C_*(X) \otimes C_*(X),$$

so called Alexander-Whitney diagonal, which turns $C_*(X)$ into a dg coalgebra. This diagonal is defined by

$$\Delta(v_{i_0}, ..., v_{i_n}) = \sum_{k=0}^{n} (v_{i_0}, ..., v_{i_k}) \otimes (v_{i_k}, ..., v_{i_n}).$$

3.8 Cohomology Algebra

The Alexander-Whitney diagonal of $C_*(V)$ induces the cup product on the dual cochain complex

$$\smile : C^*(V) \otimes C^*(V) \to C^*(V)$$

which for $\phi \in C^p(V),\ \psi \in C^q(V)$ looks as

$$\phi \smile \psi(v_{i_0}, ..., v_{i_{p+q}}) = \phi(v_{i_0}, ..., v_{i_p}) \cdot \psi(v_{i_p}, ..., v_{i_{p+q}}).$$

This structure induces on the cohomology $H^*(V)$ a structure of graded algebra.

Cohomology algebra $H^*(V)$ is more powerful invariant then cohomology groups.

Fore example two spaces $X = S^1 \times S^1$ and $Y = S^1 \vee S^1 \vee S^2$ have the same cohomology groups

$$H^0 = R, \quad H^1 = R \cdot a \oplus R \cdot b, \quad H^2 = R \cdot c,$$

with generators a, b in dimension 1 and c in dimension 2. But they have different cohomology algebras: $a \cdot b = 0$ in $H^*(Y)$ and $a \cdot b = c$ in $H^*(X)$.

4 Bar and Cobar Functors

Here we describe adjoint functors

$$B : DGAlg \rightleftarrows DGCoalg : \Omega,$$

the bar functor $B : DGAlg \to DGCoalg$ from the category of dg algebras to the category of gd coalgebras, and the cobar functor $\Omega : DGCoalg \to DGAlg$ of opposite direction.

4.1 Free Objects

4.1.1 Tensor Algebra

Let $V = \{V_i\}$ be a graded R-module. The tensor algebra generated by V is defined as

$$T(V) = R \oplus V \oplus V \otimes V \oplus V \otimes V \otimes V \oplus ... = \sum_{i=0}^{\infty} V^{\otimes i}$$

with grading $dim(a_1 \otimes ... \otimes a_m) = dim\, a_1 + ... + dim\, a_m$, and with multiplication

$$(a_1 \otimes ... \otimes a_m) \cdot (a_{m+1} \otimes ... \otimes a_{m+n}) = a_1 \otimes ... \otimes a_{m+n}.$$

The unit element for this multiplication is $1 \in R = V^{\otimes 0}$.

By i_k we denote the clear inclusion $i_k : V^{\otimes k} \to T(V)$.

Universal Property of $T(V)$

Tensor algebra $T(V)$ is the free object in the category of graded algebras: for an arbitrary graded algebra A and a map of graded modules $\alpha : V \to A$

there exists unique morphism of graded algebras $f_\alpha : T(V) \to A$ such that $f_\alpha(v) = \alpha(v)$ (i.e. $f_\alpha i_1 = \alpha$).

This morphism f_α (which is called multiplicative extension of α) is defied as $f_\alpha(a_1 \otimes ...a_m) = \alpha(a_1) \cdot ... \cdot \alpha(a_m)$. Or, equivalently f_α is described by:

$$f_\alpha i_k = \sum_k \mu^k(\alpha \otimes ... \otimes \alpha)$$

where $\mu^k : A^{\otimes k} \to A$ is the k-fold iteration of the multiplication $\mu : A \otimes A \to A$, i.e. $\mu^1 = id,\ \mu^2 = \mu,\ \mu^k = \mu(\mu^{k-1} \otimes id)$.

So, to summarize, we have the following universal property

$$\begin{array}{ccl} V & \xrightarrow{i_1} & T(V) \\ \alpha & \searrow & \downarrow f_\alpha = \sum_k \mu^k(\alpha \otimes ... \otimes \alpha) \\ & & A. \end{array}$$

Universal Property for Derivations

Tensor algebra has analogous universal property also for derivations: for a homomorphism $\beta : V \to T(V)$ of degree k there exists unique k-derivation (i.e. (id, id)-derivation)

$$D_\beta : T(V) \to T(V)$$

such that $D(v) = \beta(v)$, i.e. commutes the diagram

$$\begin{array}{ccl} V & \xrightarrow{i_1} & T(V) \\ \beta & \searrow & \downarrow D_\beta \\ & & T(V). \end{array}$$

The derivation D_β is defied as

$$D_\beta(a_1 \otimes ... \otimes a_n) = \sum_{k=1}^{n} a_1 \otimes ... \otimes a_{k-1} \otimes \beta(a_k) \otimes a_{k+1} \otimes ... \otimes a_n.$$

Or, equivalently $D_\beta \cdot i_n = \sum_k id^{\otimes(k-1)} \otimes \beta \otimes id^{\otimes(n-k)}$.

4.1.2 Tensor Coalgebra

Here we dualize the material of the previous section.

Let $V = \{V_i\}$ be a graded R-module. The tensor coalgebra cogenerated by V is defined as

$$T^c(V) = R \oplus V \oplus V \otimes V \oplus V \otimes V \otimes V \oplus ... = \sum_{i=0}^{\infty} V^{\otimes i}$$

with grading $dim(a_1 \otimes ... \otimes a_m) = dim\ a_1 + ... + dim\ a_m$, and with comultiplication $\Delta : T^c(V) \to T^c(V) \otimes T^c(V)$ given by

$$\Delta(a_1 \otimes ... \otimes a_n) = \sum_{i=0}^{n} (a_1 \otimes ... \otimes a_i) \otimes (a_{i+1} \otimes ... \otimes a_n),$$

here $(\ ) = 1 \in R = V^{\otimes 0}$.

By p_k we denote the clear projection $p_k : T^c(V) \to V^{\otimes k}$.

Universal Property of $T^c(V)$

In order to speak about universal property in this case we have to introduce some dimensional restrictions in this case.

Let $V = \{..., 0, 0, V_1, V_2, ...\}$ be a *connected* graded module, that is $V_i = 0$ for $i \leq 0$.

The tensor coalgebra of such V is the cofree object in the category of connected graded coalgebras: for a map of graded modules $\alpha : C \to V$ there exists unique morphism of graded coalgebras $f_\alpha : C \to T^c(V)$ such that $p_1 f_\alpha = \alpha$, i.e. commutes the diagram

$$\begin{array}{ccc} V & \overset{p_1}{\longleftarrow} & T^c(V) \\ \alpha & \nwarrow & \uparrow f_\alpha \\ & & C. \end{array}$$

The coalgebra map f_α (which is called comultiplicative coextension of α) is defied as $f_\alpha = \sum_k (\alpha \otimes ... \otimes \alpha)\Delta^k$, where $\Delta^k : C \to C^{\otimes k}$ is the k-th iteration of the comultiplication $\Delta : C \to C \otimes C$, i.e. $\Delta^1 = id$, $\Delta^2 = \Delta$, $\Delta^k = (\Delta^{k-1} \otimes id)\Delta$.

Universal Property for Coderivations

Tensor coalgebra has similar universal property also for coderivations, i.e. maps $\partial : C \to C'$ satisfying $\Delta\partial = (\partial \otimes id + id \otimes \partial)\Delta$.

Namely for each homomorphism $\beta : T^c(V) \to V$ there exists unique coderivation $\partial_\beta : T^c(V) \to T^c(V)$ such that $p_1\partial_\beta = \beta$, i.e. commutes the diagram

$$\begin{array}{ccc} V & \xleftarrow{p_1} & T^c(V) \\ \beta & \nwarrow & \uparrow \partial_\beta \\ & & T^c(V). \end{array}$$

The coderivation ∂_β is defied as $\partial_\beta = \sum_{k,i}(id \otimes \beta \otimes id)\Delta^3$.

More detailed: a homomorphism β in fact is a collection of homomorphisms

$$\{\beta_i : V^{\otimes\, i} \to V, \quad i = 1, 2, 3, ...\},$$

and the coderivation ∂_β is given by

$$\partial_\beta(v_1 \otimes ... \otimes v_n) = \sum_i \sum_k v_1 \otimes ... \otimes v_k \otimes \beta_i(v_{k+1} \otimes ... \otimes v_{k+i}) \otimes ... \otimes v_n \ .$$

4.2 Bar and Cobar Functors

4.2.1 Cobar Construction

Let (C, d, Δ) be a dg coalgebra with $C_i = 0,\ i \leq 1$ and let $s^{-1}C$ be the desuspension of C, that is $(sA)_k = A_{k+1}$.

The cobar construction ΩC is defined as tensor algebra $T(s^{-1}C)$. We use the following notation for elements of this tensor coalgebra

$$s^{-1}a_1 \otimes ... \otimes s^{-1}a_n = [a_1, ..., a_n].$$

So the dimension of $[a_1, ..., a_n]$ is $\sum_i\ dim\ a_i - n$.

The differential $d_{\Omega C} : \Omega C \to \Omega C$ is defined as

$$d_{\Omega C}[a_1, ..., a_n] =$$
$$\textstyle\sum_i \pm[a_1, ..., a_{i-1}, da_i, a_{i+1}, ..., a_n] + \sum_i \pm[a_1, ..., a_{i-1}, \Delta(a_i), a_{i+1}, ..., a_n].$$

In fact this a derivation defined by the above universal property for

$$\beta[a] = da + \Delta a,$$

thus $d_{\Omega C}$ is a derivation. Besides $d_{\Omega C}d_{\Omega C} = 0$, so $\Omega C \in DGAlg$.

4.2.2 Bar Construction

Let $(A, d, \cdot)$ be a dg algebra with $A_i = 0,\ i \leq 1$ and let sA be the suspension of A, that is $(sA)_k = A_{k-1}$.

The bar construction BA is defined as tensor coalgebra $T^c(sA)$. We use the following notation for elements of this tensor coalgebra

$$sa_1 \otimes ... \otimes sa_n = [a_1, ..., a_n].$$

So the dimension of $[a_1, ..., a_n]$ is $\sum_i \ dim\ a_i + n$.

The differential $d_B : BA \to BA$ is defined as

$$d_B[a_1, ..., a_n] = \sum_i \pm[a_1, ..., a_{i-1}, da_i, a_{i+1}, ..., a_n] + \sum_i \pm[a_1, ..., a_i \cdot a_{i+1}, ..., a_n].$$

In fact this the coderivation defined by the above universal property for

$$\beta[a_1, ..., a_n] = \begin{cases} [\ da_1] & for \quad n = 1; \\ [\ a_1 \cdot a_2\ ] & for \quad n = 2 \\ 0 & for \quad n > 2. \end{cases}$$

Thus d_B is a coderivation. Besides $d_B d_B = 0$, so $BA \in DGCoalg$.

4.2.3 Adjunction

Let (C, d, Δ) be a dgc, (A, d, μ) a dga. A twisting cochain [3] is a homomorphism $\tau : C \to A$ of degree $+1$ satisfying the Browns' condition

$$d\tau + \tau d = \tau \smile \tau, \tag{1}$$

where $\tau \smile \tau' = \mu_A(\tau \otimes \tau')\Delta$. We denote by $T(C, A)$ the set of all twisting cochains $\tau : C \to A$.

There are universal twisting cochains $C \to \Omega C$ and $BA \to A$ being clear inclusion and projection respectively. Here are essential consequences of the condition (1):

(i) The multiplicative extension $f_\tau : \Omega C \to A$ is a map of dg algebras, so there is a bijection $T(C, A) \leftrightarrow Hom_{DG-Alg}(\Omega C, A)$;

(ii) The comultiplicative coextension $f_\tau : C \to BA$ is a map of dg coalgebras, so there is a bijection $T(C, A) \leftrightarrow Hom_{DG-Coalg}(C, BA)$.

Thus we have two bijections

$$Hom_{DG-Alg}(\Omega C, A) \leftrightarrow T(C, A) \leftrightarrow Hom_{DG-Coalg}(C, BA).$$

Besides, there are two week equivalences (homology isomorphisms)

$$\alpha_A : \Omega B(A) \to A, \qquad \beta_C : C \to B\Omega(C).$$

5 Homotopy Algebras - Lack of Associativity

This is the general title for dg algebras for which some classical axioms of algebras are satisfied not strictly but just up to chain homotopy: for example there are notions of *homotopy associative, homotopy commutative, homotopy Lie,* etc. dg algebras.

How such homotopy algebra show up? Suppose A is a dg algebra of some sort which satisfies some classical axioms, say associativity or/and commutativity, and B is a chain complex chain equivalent to A. Then often it is possible to transport the structure from A to B, but usually these axioms brake up, but they are satisfied up to (higher) homotopy.

In this section we present the notion of A_∞-algebra, which is *strong homotopy associative* algebra, and its commutative version: the notion of C_∞-algebra.

In the forthcoming sections we discuss also the notion of homotopy G-algebra, which is sort of *strong homotopy commutative* algebra.

5.1 A_∞ algebras

The notion of A_∞-algebra was introduces by J. Stasheff [31] . This notion generalizes the notion of differential graded algebra.

Definition 10 *An A_∞-algebra is a graded module $M = \{M^k\}_{k\in Z}$ equipped with a sequence of operations*

$$\{m_i : M \otimes ...(i-times)... \otimes M \to M, i = 1, 2, 3, ...\}$$

satisfying the conditions $m_i((\otimes^i M)^q) \subset M^{q-i+2}$, that is deg $m_i = 2 - i$, and

$$\sum_{k=0}^{i-1} \sum_{j=1}^{i-k} \pm m_{i-j+1}(a_1 \otimes ... \otimes a_k \otimes m_j(a_{k+1} \otimes ... \otimes a_{k+j}) \otimes ... \otimes a_i) = 0. \tag{2}$$

For $i = 1$ this condition reads

$$m_1 m_1 = 0.$$

For $i = 2$ this condition reads

$$m_1 m_2(a_1 \otimes a_2) \pm m_2(m_1(a_1) \otimes a_2) \pm m_2(a_1 \otimes m_1(a_2)) = 0.$$

For $i = 3$ this condition reads

$$\begin{gathered} m_1 m_3(a_1 \otimes a_2 \otimes a_3) \pm \\ m_3(m_1(a_1) \otimes a_2 \otimes a_3) \pm m_3(a_1 \otimes m_1(a_2) \otimes a_3) \pm m_3(a_1 \otimes a_2 \otimes m_1(a_3)) \pm \\ m_2(m_2(a_1 \otimes a_2) \otimes a_3) \pm m_2(a_1 \otimes m_2(a_2 \otimes a_3) = 0. \end{gathered}$$

These three condition mean that for an A_∞-algebra $(M, \{m_i\})$ first two operations form a *nonassociative* dga (M, m_1, m_2) with differential m_1 and multiplication m_2 which is associative just up to homotopy and the suitable homotopy is the operation m_3.

Definition 11 *A morphism of A_∞-algebras*

$$\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$$

is a sequence $\{f_i : \otimes^i M \to M', i = 1, 2, ..., \ deg\ f_1 = 1 - i\}$ such that

$$\begin{array}{l} \sum_{k=0}^{i-1} \sum_{j=1}^{i-k} \pm \\ f_{i-j+1}(a_1 \otimes ... \otimes a_k \otimes m_j(a_{k+1} \otimes ... \otimes a_{k+j}) \otimes ... \otimes a_i) = \\ \sum_{t=1}^{i} \sum_{k_1+...+k_t=i} \pm \\ m'_t(f_{k_1}(a_1 \otimes ... \otimes a_{k_1}) \otimes ... \otimes f_{k_t}(a_{i-k_t+1} \otimes ... \otimes a_i)). \end{array} \tag{3}$$

The composition of A_∞ morphisms

$$\{h_i\} : (M, \{m_i\}) \xrightarrow{\{f_i\}} (M', \{m'_i\}) \xrightarrow{\{g_i\}} (M'', \{m''_i\})$$

is defined as

$$\begin{array}{l} h_n(a_1 \otimes ... \otimes a_n) = \sum_{t=1}^{n} \sum_{k_1+...+k_t=n} \\ g_n(f_{k_1}(a_1 \otimes ... \otimes a_{k_1}) \otimes ... \otimes f_{k_t}(a_{n-k_t+1} \otimes ... \otimes a_n). \end{array} \tag{4}$$

The bar construction argument (see (5.1.1) bellow) allows to show that so defined composition satisfies the condition(3).

For a morphism $\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ the first component $f_1 : (M, m_1) \to (M', m'_1)$ is a chain map which is *multiplicative* just up to homotopy and the suitable homotopy is the map f_2.

A_∞ algebra of type $(M, \{m_1, m_2, 0, 0, ...\})$ is a dga with the differential m_1 and strictly associative multiplication m_2. Furthermore, a morphism of such A_∞-algebras of type $\{f_1, 0, 0, ...\}$ is a strictly multiplicative chain map. Thus the category of dg algebras is the subcategory of the category of A_∞-algebras.

5.1.1 Bar construction of an A_∞-algebra

Let $(M, \{m_i\})$ be an A_∞-algebra. The structure maps m_i define the map $\beta : T^c(s^{-1}M) \to s^{-1}M$ by $\beta[a_1, ..., a_n] = [s^{-1}m_n(a_1 \otimes ... \otimes a_n)]$. Extending this β as a coderivation we obtain $d_\beta : T^c(s^{-1}M) \to T^c(s^{-1}M)$ which in fact looks as

$$d_\beta[a_1, ..., a_n] = \sum_k \pm[a_1, ..., a_k, m_j(a_{k+1} \otimes ... \otimes a_{k+j}), a_{k+j+1}, ...a_n].$$

The defining condition (2) of A_∞-algebra guarantees that $d_\beta d_\beta = 0$. The obtained dg coalgebra $(T^c(s^{-1}M), d_\beta, \Delta)$ is called *bar construction* of A_∞-algebra $(M, \{m_i\})$ and is denoted by $\tilde{B}(M)$.

For an A_∞-algebra of type $(M, \{m_1, m_2, 0, 0, ...\})$ this bar construction coincides with the ordinary bar construction of this dga.

A morphism of A_∞-algebras $\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ defines a dg coalgebra map of bar constructions

$$F = \tilde{B}(\{f_i\}) : \tilde{B}(M, \{m_i\}) \to \tilde{B}(M', \{m'_i\})$$

as follows: $\{f_i\}$ defines the map $\alpha : T^c(s^{-1}M) \to s^{-1}M$ by $\alpha[a_1, ..., a_n] = [s^{-1}f_n(a_1 \otimes ... \otimes a_n)]$. Extending this α as a coalgebra map we obtain $F : T^c(s^{-1}M) \to T^c(s^{-1}M)$ which in fact looks as

$$F[a_1, ..., a_n] = \sum \pm[f_{k_1}(a_1 \otimes ... \otimes a_{k_1}), ..., f_{k_t}(a_{n-k_t+1} \otimes ... \otimes a_n)].$$

The defining condition (3) of A_∞ morphism guarantees that F is a chain map.

Now we are able to show that the composition of A_∞ morphisms is correctly defined: to the composition of morphisms (4) corresponds the composition of dg coalgebra maps

$$\tilde{B}((M, \{m_i\})) \xrightarrow{\tilde{B}(\{f_i\})} \tilde{B}((M', \{m'_i\})) \xrightarrow{\tilde{B}(\{g_i\})} \tilde{B}((M'', \{m''_i\}))$$

which is a dg coalgebra map, thus for the projection $p_1\tilde{B}(\{g_i\})\tilde{B}(\{f_i\})$, i.e. for the collection $\{h_i\}$, the condition (3) is satisfied.

5.2 C_∞ algebras

This is the commutative version of the notion of A_∞-algebra. For an A_∞-algebra $(M, \{m_i\})$ it is clear how to say that the operation $m_2 : M \otimes M \to M$ is commutative, but what about the commutativity of higher operations $m_i : M \otimes ... \otimes M \to M$? We are going to describe this now.

5.2.1 The notion of C_∞-algebra

In fact $T(V)$ and $T^c(V)$ coincide as graded modules, but the multiplication of $T(V)$ and the comultiplication of $T^c(V)$ are not compatible with each other, so they do not define a graded bialgebra structure on $T(V) = T^c(V)$.

Nevertheless there exists the *shuffle* multiplication $\mu_{sh} : T^c(V) \otimes T^c(V) \to T^c(V)$ introduced by Eilenberg and MacLane which turns $(T^c(V), \Delta, \mu_{sh})$ into a graded bialgebra.

This multiplication is defined as a graded coalgebra map induced by the universal property of $T^c(V)$ by $\alpha : T^c(V) \otimes T^c(V) \to V$ given by $\alpha(v \otimes 1) = \alpha(1 \otimes v) = v$ and $\alpha = 0$ otherwise. This multiplication is associative and in fact is given by

$$\mu_{sh}([a_1, ..., a_m] \otimes [a_{i+1}, ..., a_n]) = \sum \pm [a_{\sigma(1)}, ..., a_{\sigma(n)}]),$$

where the summation is taken over all (m, n)-shuffles. That is over all permutations of the set $(1, 2, ..., n + m)$ which satisfy the condition: $i < j$ if $1 \leq \sigma(i) < \sigma(j) \leq n$ or $n + 1 \leq \sigma(i) < \sigma(j) \leq n + m$. In particular

$$[a] *_{sh} [b] = [a, b] \pm [b, a],$$

$$[a] *_{sh} [b, c] = [a, b, c] \pm [b, a, c] \pm [b, c, a].$$

Now we can define the notion of C_∞-algbera, which is a commutative version of the notion of A_∞-algebra.

Definition 12 *([28], [15],[23], [11]) A C_∞-algebra is an A_∞-algebra $(M, \{m_i\})$ which additionally satisfies the following condition: each operation m_i vanishes on shuffles, that is for $a_1, ..., a_i \in M$ and $k = 1, 2, ..., i - 1$*

$$m_i(\mu_{sh}((a_1 \otimes ... \otimes a_k) \otimes (a_{k+1} \otimes ... \otimes a_i))) = 0. \tag{5}$$

Definition 13 *A morphism of C_∞-algebras is defined as a morphism of A_∞-algebras $\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ whose components f_i vanish on shuffles, that is*

$$f_i((\mu_{sh}(a_1 \otimes ... \otimes a_k) \otimes (a_{k+1} \otimes ... \otimes a_i))) = 0. \quad (6)$$

The composition is defined as in A_∞ case and the bar construction argument (see (5.1.1) bellow) allows to show that the composition is a C_∞ morphism.

In particular for the operation m_2 we have $m_2(a \otimes b \pm b \otimes a) = 0$, so a C_∞-algebra of type $(M, \{m_1, m_2, 0, 0, ...\})$ is a commutative dg algebra (cdga) with the differential m_1 and strictly associative and commutative multiplication m_2. Thus the category of cdg algebras is the subcategory of the category of C_∞-algebras.

5.2.2 Bar construction of a C_∞-algebra

The notion of C_∞-algebra is motivated by the following observation. If a dg algebra (A, d, μ) is graded commutative then the differential of the bar construction BA is not only a coderivation but also a derivation with respect to the shuffle product, so the bar construction $(BA, d_\beta, \Delta, \mu_{sh})$ of a cdga is a dg bialgebra.

By definition the bar construction of an A_∞-algebra $(M, \{m_i\})$ is a dg coalgebra $\tilde{B}(M) = (T^c(s^{-1}M), d_\beta, \Delta)$.

But if $(M, \{m_i\})$ is a C_∞-algebra, then $\tilde{B}(M)$ becomes a dg bialgebra:

Proposition 13 *For an A_∞-algebra $(M, \{m_i\})$ the differential of the bar construction d_β is a derivation with respect to the shuffle product if and only if each operation m_i vanishes on shuffles, that is $(M, \{m_i\})$ is a C_∞-algebra.*

Proof. The map $\Phi : T^c(s^{-1}M) \otimes T^c(s^{-1}M) \to T^c(s^{-1}M)$ defined as $\Phi = d_\beta \mu_{sh} - \mu_{sh}(d_\beta \otimes id + id \otimes d_\beta)$ is a coderivation. Thus, according to universal property of $T^c(s^{-1}M)$ the map Φ is trivial if and only if $p_1\Phi = 0$ and the last condition means exactly (5).

Proposition 14 *Let $\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ be an A_∞-algebra morphism of C_∞-algebras. Then the induced map of bar constructions $\tilde{B}\{f_i\}$ is a map of dg bialgebras if and only if each f_i vanishes on shuffles, that is $\{f_i\}$ is a morphism of C_∞-algebras.*

Proof. The map $\Psi = \tilde{B}\{f_i\}\mu_{sh} - \mu_{sh}(\tilde{B}\{f_i\} \otimes \tilde{B}\{f_i\})$ is a coderivation. Thus, according to universal property of $T^c(s^{-1}M)$ the map Ψ is trivial if and only if $p_1\Psi = 0$ and the last condition means exactly (6).

Thus the bar functor maps the subcategory of C_∞-algebras to the category of dg bialgebras.

5.3 Minimality

Let $\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ be a morphism of A_∞-algebras. It follows from (3) that the first component $f_1 : (M, m_1) \to (M', m'_1)$ is a chain map.

A weak equivalence of A_∞-algebras is defined as a morphism $\{f_i\}$ for which $B(\{f_i\})$ is a weak equivalence (homology isomorphism) of dg coalgebras. The standard spectral sequence argument allows to prove the following

Proposition 15 *A morphism of A_∞-algebras is a weak equivalence if and only if it's first component $f_1 : (M, m_1) \to (M', m'_1)$ is a weak equivalence of chain complexes.*

Proposition 16 *A morphism of A_∞-algebras is an isomorphism if and only if it's first component $f_1 : (M, m_1) \to (M', m'_1)$ is an isomorphism.*

Proof. The components of opposite morphism $\{g_i\} : (M', \{m'_i\}) \to (M, \{m_i\})$ can be solved inductively from the equation $\{g_i\}\{f_i\} = \{id_M, 0, 0, ...\}$.

Definition 14 *An A_∞-algebra $(M, \{m_i\})$ we call minimal if $m_1 = 0$.*

In this case (M, m_2) is *strictly* associative graded algebra.

From the above propositions easily follows

Proposition 17 *Each weak equivalence of minimal A_∞-algebras is an isomorphism.*

It is clear that all above is true for C_∞-algebras, thus

Proposition 18 *Each weak equivalence of minimal C_∞-algebras is an isomorphism.*

Definition 15 *A minimal A_∞-algebra (C_∞-algebra) $(M, \{m_i\})$ we call degenerate if it is isomorphic in the category of A_∞ (C_∞) algebras to the graded (commutative) algebra (M, m_2).*

5.4 A_∞-algebra structure in homology

Let (A, d, μ) be a dg algebra and $(H(A), \mu^*)$ be it's homology algebra. Although the product in $H(A)$ is associative, there appears a structure of a (generally nondegenerate) minimal A_∞-algebra, which can be considered as an A_∞ *deformation* of $(H(A), \mu^*)$, [21] . Namely, in [13], [14] the following result was proved (see also [28], [12]):

Theorem 1 *Suppose for a dg algebra A all homology modules $H^i(A)$ are free.*

Then there exist: a structure of minimal A_∞-algebra $(H(A), \{m_i\})$ on $H(A)$ and a weak equivalence of A_∞-algebras

$$\{f_i\} : (H(A), \{m_i\}) \to (A, \{d, \mu, 0, 0, ...\})$$

such, that $m_1 = 0$, $m_2 = \mu^*$, $f_1^* = id_{H(A)}$.

Furthermore, for a dga map $f : A \to A'$ there exists a morphism of A_∞-algebras $\{f_i\} : (H(A), \{m_i\}) \to (H(A'), \{m'_i\})$ with $f_1 = f^*$.

Such a structure is unique up to isomorphism in the category of A_∞-algebras: if $(H(A), \{m_i\})$ and $(H(A), \{m'_i\})$ are two such A_∞-algebra structures on $H(A)$ then for $id : A \to A$ there exists $\{f_i\} : (H(A), \{m_i\}) \to (H(A), \{m'_i\})$ with $f_1 = id$, so, since of Proposition 16 $\{f_i\}$ is an isomorphism.

Let us look at the first new operation $m_3 : H(A)\otimes H(A)\otimes H(A) \to H(A)$. Let $f_1 : H(A) \to A$ be a cycle-choosing homomorphism: $f_1(a) \in a \in H(A)$. This map is not multiplicative but $f_1(a \cdot b) - f_1(a) \cdot f(b) \sim 0 \in C$ so there exists $f_2 : H(A) \otimes H(A) \to A$ s.t. $f_1(a \cdot b) - f_1(a) \cdot f(b) = \partial f_2(a \otimes b)$. We define $m_3(a \otimes b \otimes c) \in H(A)$ as the homology class of the cycle

$$f_1(a) \cdot f_2(b \otimes c) \pm f_2(a \cdot b \otimes c) \pm f_2(a \otimes b \cdot c) \pm f_2(a \otimes b) \cdot f_1(c).$$

From this description immediately follows the connection of m_3 with Massey product: If $a, b, c \in H(A)$ is a Massey triple, i.e. if $a \cdot b = b \cdot c = 0$, then $m_3(a\otimes b\otimes c)$ belongs to the Massey product $< a, b, c >$. This gives examples of dg algebras with essentially nontrivial homology A_∞-algebras.

5.4.1 Main examples and applications

Taking $A = C^*(X)$, the cochain dg algebra of a 1-connected space X, we obtain an A_∞-algebra structure $(H^*(X), \{m_i\})$ on cohomology algebra $H^*(X)$.

Cohomology algebra equipped with this additional structure carries more information then just the cohomology algebra. Some applications of this structure are given in [14] , [18]. For example the cohomology A_∞-algebra $(H^*(X), \{m_i\})$ determines cohomology of the loop space $H^*(\Omega X)$ when just the algebra $(H^*(X), m_2)$ does not:

Theorem 2 $H(\tilde{B}(H^*(X), \{m_i\})) = H^*(\Omega X)$.

Taking $A = C_*(G)$, the chain dg algebra of a topological group G, we obtain an A_∞-algebra structure $(H_*(G), \{m_i\})$ on the Pontriagin algebra $H_*(G)$. The homology A_∞-algebra $(H_*(G), \{m_i\})$ determines homology of the classifying space $H_*(B_G)$ when just the Pontriagin algebra $(H_*(G), m_2)$ does not:

Theorem 3 $H(B(\tilde{H_*}(G), \{m_i\})) = H_*(B_G)$.

5.5 C_∞-algebra structure in homology of a commutative dg algebra

There is a commutative version of the above main theorem, see[17], [18], [23]:

Theorem 4 *Suppose for a commutative dg algebra A all homology R-modules $H^i(A)$ are free.*

Then there exist: a structure of minimal C_∞-algebra $(H(A), \{m_i\})$ on $H(A)$ and a weak equivalence of C_∞-algebras

$$\{f_i\} : (H(A), \{m_i\}) \to (A, \{d, \mu, 0, 0, ...\})$$

such, that $m_1 = 0$, $m_2 = \mu^*$, $f_1^* = id_{H(A)}$.

Furthermore, for a cdga map $f : A \to A'$ there exists a morphism of C_∞-algebras $\{f_i\} : (H(A)\{m_i\}) \to (H(A')\{m'_i\})$ with $f_1 = f^*$.

Such a structure is unique up to isomorphism in the category of C_∞-algebras.

Bellow we present some applications of this C_∞-algebra structure in rational homotopy theory.

5.5.1 Applications in Rational Homotopy Theory

Let X be a 1-connected space. In the case of rational coefficients there exist Sullivan's *commutative* cochain complex $A(X)$ of X. It is well known that the weak equivalence type of cdg algebra $A(X)$ determines the rational homotopy type of X: 1-connected X and Y are *rationally homotopy equivalent* if and only if $A(X)$ and $A(Y)$ are weekly homotopy equivalent cdg algebras. Indeed, in this case $A(X)$ and $A(Y)$ have *isomorphic* minimal models $M_X \approx M_Y$, and this implies that X and Y are rationally homotopy equivalent. This is the key geometrical result of Sullivan which we are going to exploit bellow.

Now we take $A = A(X)$ and apply the Theorem 4. Then we obtain on $H(A) = H^*(X, Q)$ a structure of minimal C_∞ algebra $(H^*(X, Q), \{m_i\})$ which we call *rational cohomology C_∞-algebra of X*.

Generally isomorphism of rational cohomology algebras $H^*(X, Q)$ and $H^*(Y, Q)$ does not imply homotopy equivalence $X \sim Y$ even rationally. We claim that $(H^*(X, Q), \{m_i\})$ is *complete* rational homotopy invariant:

Theorem 5 *1-connected X and X' are rationally homotopy equivalent if and only if $(H^*(X, Q), \{m_i\})$ and $(H^*(X', Q), \{m'_i\})$ are isomorphic as C_∞-algebras.*

This theorem in fact classifies rational homotopy types with given cohomology algebra H as all possible minimal C_∞-algebra structures on H modulo C_∞ isomorphisms.

6 Homotopy Algebras - Lack of Commutativity

6.1 Commutativity

A dg algebra (A, d, μ) is called *commutative* if $a \cdot b = (-1)^{|a||b|} b \cdot a$, that is commutes the diagram

$$\begin{array}{ccc} A \otimes A & \xrightarrow{T} & A \otimes A \\ \mu \searrow & & \swarrow \mu \\ & A, & \end{array}$$

where $T(a \otimes b) = (-1)^{|a||b|}(b \otimes a)$ is the map which interchanges the factors.

Of course if (A, d, μ) is a cdg algebra (commutative dg algebra) then its homology $(H(A), \mu^*)$ is a commutative graded algebra.

An example of cgd algebra is DeRham complex.

A dg coalgebra (C, d, Δ) is called cocommutative if $T\Delta = \Delta$, i.e. commutes the diagram

$$\begin{array}{ccccc} C \otimes C & & \xrightarrow{T} & & C \otimes C \\ & \Delta \nwarrow & & \nearrow \Delta & \\ & & C. & & \end{array}$$

Proposition 19 *If (C, d, Δ) is a cocommutative dg coalgebra and (A, d, μ) is commutative dg algebra then $(Hom(C, A), D, \smile)$ is a commutative dg algebra.*

Particularly, if a chain complex (C, d, Δ) is a cocommutative dg coalgebra and R is a commutative algebra then the dual chain cochain complex $Hom(C, R)$ is commutative dg algebra.

6.2 Commutativity Up to Homotopy

Usual situation is that a dg algebra (A, d, μ) is not strictly commutative but its homology algebra $H(A)$ is.

This happens when (A, d, μ) is *commutative up to homotopy*, that is the chain maps $\mu, \mu T : A \otimes A \to A$ are chain homotopic, i.e. there exists chain homotopy $h : A \otimes A \to A$ such that

$$ab - (-1)^{|a||b|} ba = dh(a \otimes b) + h(da \otimes b + (-1)^{|a|} a \otimes db).$$

Let us use the standard notation for such homotopy $h(a \otimes b) = a \smile_1 b$ (Steenrod's $\smile_1$-product).

In this notation the above condition looks

$$d(a \smile_1 b) = -da \smile_1 b - (-1)^{|a|} a \smile_1 db + a \cdot b - (-1)^{|a||b|} b \cdot a \qquad (7)$$

(Steenrod condition).

The existence in a dg algebra (A, d, μ) of a $\smile_1$-product implies homotopy of μ and μT, thus the induced homology maps

$$H(\mu), H(\mu T) : H(A \otimes A) \to H(A)$$

coincide. Composing with standard $H(A) \otimes H(A) \to H(A \otimes A)$ we obtain

$$\mu^* = \mu^* T : H(A) \otimes H(A) \to H(A \otimes A) \to H(A).$$

Thus we have the

Proposition 20 *If a dg algebra (A, d, μ) is additionally equipped with a $\smile_1$-product satisfying the condition (7) then $H(A)$ is commutative.*

6.3 Steenrod's Geometric $\smile_1$-product

Back to our ordered simplicial complex V. In the chain complex $C_*(V)$ there is the following diagonal

$$\Delta_1 : C_*(V) \to C_*(V) \otimes C_*(V)$$

of degree 1

$$\Delta_1(v_{i_0}, ..., v_{i_k}, ..., v_{i_l}, ..., v_{i_n}) = \sum_{0 \le k < l \le n} (v_{i_0}, ..., v_{i_k}, v_{i_l}, ..., v_{i_n}) \otimes (v_{i_k}, ..., v_{i_l}).$$

This cooperation satisfies the condition

$$\Delta_1 d + (d \otimes id + id \otimes d)\Delta_1 = \Delta - T\Delta,$$

and this condition implies that the operation of degree -1

$$\smile_1 : C^*(V) \otimes C^*(V) \to C^{*-1}(V)$$

dual to Δ_1, i.e. defined as

$$\phi \smile_1 \psi = \mu(\phi \otimes \psi)\Delta_1$$

satisfies the Steenrod's condition (7).

6.4 Steenrod's $\smile_i$-products

The $\smile_1$-product is part of sequence of Steenrod $\smile_i$, $i = 0, 1, 2, ...$ operations. Each operation $\smile_i$ is a homomorphism of degree $-i$

$$\smile_i : C \otimes C \to C,$$

which satisfies the condition

$$d(a \smile_i b) = \\ (-1)^i da \smile_i b + (-1)^{i+|a|} a \smile_i db - (-1)^i a \smile_{i-1} b - (-1)^{|a||b|} b \smile_{i-1} a, \tag{8}$$

so the $\smile_i$-product measures the deviation from commutativity of the $\smile_{i-1}$ product.

Since $a \smile_i b \in C^{|a|+|b|-i}$ we see that $a \smile_i b = 0$ for $i > |a| + |b|$.

6.5 Steenrod's Geometric $\smile_i$-products on $C_*(V)$

The $\smile_i$-product is dual to the cooperation

$$\Delta_i : C_*(V) \to C_*(V) \otimes C_*(V)$$

of degree i which is hard to describe directly. Bellow we give the description of Δ_i in operadic terms. Here is particular expression for Δ_2:

$$\Delta_2(v_{i_0}, ..., v_{k_1}, ..., v_{k_2}, ..., v_{k_3}, ..., v_{i_n}) = \\ \sum_{0 \le k_1 < k_2 < k_3 \le n} (v_{i_0}, ..., v_{i_{k_1}}, v_{i_{k_2}}, ..., v_{i_{k_3}}) \otimes (v_{i_{k_1}}, ..., v_{i_{k_2}}, v_{i_{k_3}}, ..., v_{i_n}).$$

The dual $\smile_i$ product is defined as

$$\phi \smile_i \psi = \mu(\phi \otimes \psi)\Delta_i$$

and it satisfies the condition (8).

Two more properties of $\smile_i$ follow from the construction:

1. For $\phi \in C^p(V)$ one has

$$\phi \smile_p \phi = \phi$$

2. For $\phi \in C^p(V)$ and $\psi \in C^q(V)$ one has $\phi \smile_i \psi = 0$ for $i > min(p, q)$.

6.6 Steenrod Squares

Let's define a dg algebra with $\smile_i$ products as an object $(C^*, d, \{\smile_i,\ i = 0, 1, 2, ...\})$ where $(C^*, d, \smile_0)$ is a dg algebra and the operations $\smile_i$ satisfy the conditions (8). An example is $C^*(V)$.

This structure defines in mod2 case the cohomology operations

$$Sq_i : H^p(C) \to H^{2p-i}(C)$$

defined as follows.

Take any $a \in H^p(C)$ and let $z_a \in a$ be a cocycle from a. It follows from (8) that $d(z_a \smile_1 z_a) = 0$. Let define $Sq_i(a)$ as the cohomology class of this cocycle, i.e.

$$Sq_i(a) = cl(z_a \smile_1 z_a) \in H^{2p-i}(C).$$

Clear that

$$Sq_0(a) = a^2.$$

The Steenrod formula (8) allows to check that

1. This definition is correct, i.e. the class $Sq_i(a)$ does not depend on choosing of a cocycle $z_a \in a$.
2. Sq_i is a homomorphism $Sq_i(a + b) = Sq_i(a) + Sq_i(b)$.
3. Sq_i is functorial: for a chain map

$$f : (C, d, \{\smile_i\}) \to (C', d, \{\smile_i\})$$

satisfying $f(a \smile_i b) = f(a) \smile_i f(b)$ commutes the diagram

$$\begin{array}{ccccc} & H^p(C) & \overset{Sq_i}{\to} & H^{2p-i}(C) & \\ H^p(f) & \downarrow & & \downarrow & H^{2p-i}(f) \\ & H^p(C') & \overset{Sq_i}{\to} & H^{2p-i}(C') & . \end{array}$$

The Steenrod square $Sq^i : H^p(C) \to H^{p+1}(C)$ is defined as $Sq^i = Sq_{p-i}$.

We already know that $Sq^i : H^p(C) \to H^{p+i}(C)$ is a functorial homomorphism, and $Sq^i(a) = a^2$ for $a \in H^i(C)$.

There is one more important property of geometric Steenrod squares:
Cartan formula. $Sq^i(ab) = \sum_j Sq^k(a)Sq^{i-k}(b)$.

Equivalently this means that the homomorphim $Sq : H(C) \to H(C)$ is a multiplicative homomorphism $Sq(ab) = Sq(a)Sq(b)$.

The Cartan formula does not follow just from the Steenrod condition (8). The geometric Steenrod $\smile_i$ products have some more properties which we must add to the definition of the notion of *dg algebra with Steenrod* $\smile_i$ *products.*

One of such properties is so called Hirsch formula

$$(a \cdot b) \smile_1 c = a \cdot (b \smile_1 c) + (a \smile c)b$$

which we discuss next.

7 Homotopy Gerstenhaber Algebras

This is a notion of a dga with "good" $\smile_1$ product.

7.1 Hirsch Formula

Let us denote the $\smile$ product just by dot $a \smile b = a \cdot b$.

The geometric $\smile_1$ product has the following property (so called Hirsch formula)

$$(ab) \smile_1 c = a(b \smile_1 c) + (a \smile_1 c)b.$$

This means that the map $- \smile_1 c : C \to C$, is a derivation.

What about the map $a \smile - : C \to C$ i.e. when we multiply by $\smile_1$ from the left?

This map generally is not a derivation, i.e. the expression

$$a \smile (bc) - b(a \smile_1 c) - (a \smile_1 b)c$$

is not zero, but it is homotopical to zero. This means that there exists suitable chain homotopy, a map

$$E_{12} : C \otimes C \otimes C \to C$$

of degree -2 such that

$$a \smile (bc) - b(a \smile_1 c) - (a \smile_1 b)c =$$
$$dE_{12}(a; b, c) + E_{12}(da; b, c) + E_{12}(a; db, c) + E_{12}(a :, b, dc),$$

here we use the notation $E_{12}(a;b,c) = E_{12}(a\otimes b\otimes c)$.

The cochain complex $C^*(V)$ of a simplicial complex V possesses such an operation: The cochain operation E_{12} is dual to the chain cooperation

$$\Delta_{12} : C(V) \to C(V)\otimes C(V)\otimes C(V)$$

given by

$$\Delta_{12}(v_{i_0},...,v_{k_1},...,v_{k_2},...,v_{k_3},...,...,v_{k_4},...,v_{i_n}) = \\ \sum\nolimits_{0\le k_1<k_2<k_3<k_4\le n} \\ (v_{i_0},...,v_{i_{k_1}},v_{i_{k_2}},...,v_{i_{k_3}},v_{i_{k_4}},...,v_{i_n})\otimes(v_{i_{k_1}},...,v_{i_{k_2}})\otimes(v_{i_{k_3}},...,v_{i_{k_4}}).$$

So we have in the cochain complex of a topological space operation $E_{1,2}$ which controls the interaction of $\smile$ and $\smile_1$ products. Actually $E_{1,2}$ is not alone, it comes together with a sequence cochain operations $E_{1,k}$ which form on $C^*(X)$ a structure called *homotopy G-algebra*, which we introduce now.

7.2 The Notion of Homotopy G-algebra

The notion of homotopy G-algebra (hGa) was introduced in [9].

A hGa $(C,d,\cdot,\{E_{1,k}\})$ is a dga $(C,d,\cdot)$ equipped additionally by a sequence of operations

$$\{E_{1,k} : C\otimes C^{\otimes k} \to C,\ k = 1,2,...\}$$

satisfying some coherence conditions (see bellow).

In fact a hGa as an additional structure on a dg algebra $(C,d,\cdot)$ that induces a *Gerstenhaber algebra* structure on homology. The source of the defining identities and the main example was Hochschild cochain complex $C^*(A,A)$.

Another point of view is that hGa is a particular case of $B(\infty)$-algebra: this is an additional structure on a dg algebra $(C,d,\cdot)$ that induces a dg bialgebra structure on the bar construction BC.

There is the third aspect of hGa: this is a structure which measures the noncommutativity of C, since the starting operation $E_{1,1}$ of this structure is similar to the $\smile_1$ product.

There are three remarkable examples of homotopy G-algebras.

The first one is the cochain complex of 1-reduced simplicial set $C^*(X)$. The operations $E_{1,k}$ here are dual to cooperations defined by Baues in [2], and the starting operation $E_{1,1}$ is the classical Steenrod's $\smile_1$ product.

The second example is the Hochschild cochain complex $C^*(A, A)$ of an associative algebra A. The operations $E_{1,k}$ here were defined in [15] with the purpose to describe $A(\infty)$-algebras in terms of Hochschild cochains although the properties of those operations which were used as defining ones for the notion of homotopy G-algebra in [9] did not appear there. These operations were defined also in [10]. Again the starting operation $E_{1,1}$ is the classical Gerstenhaber's circle product which is sort of $\smile_1$-product in the Hochschild complex.

The third example is the the cobar construction ΩC of a dg bialgebra C. The operations $E_{1,k}$ are constructed in [20]. And again the starting operation $E_{1,1}$ is classical, it is Adams's $\smile_1$-product defined for ΩC in [1].

Now we give the explicit definition of a homotopy G-algebra.

Definition 16 *A homotopy G algebra is a differential graded algebra* $(C, d, \cdot)$ *together with a given sequence of multioperations*

$$E_{1,k} : C \otimes C^{\otimes^k} \to C, \quad k = 1, 2, 3, ... \quad ,$$

subject of the following conditions

$degE_{1,k} = -k,\ E_{1,0} = id;$

$$\begin{aligned} &dE_{1,k}(a; b_1, ..., b_k) + E_{1,k}(da; b_1, ..., b_k) + \textstyle\sum_i E_{1k}(a; b_1, ..., db_i, ..., b_k) = \\ &b_1E_{1,k-1}(a; b_2, ..., b_k) + \textstyle\sum_i E_{1,k-1}(a; b_1, ..., b_ib_{i+1}, ..., b_k)+ \\ &E_{1,k-1}(a; b_1, ..., b_{k-1})b_k; \end{aligned} \tag{9}$$

$$\begin{aligned} &a_1E_{1,k}(a_2; b_1, ..., b_k) + E_{1,k}(a_1 \cdot a_2; b_1, ..., b_k) + E_{1,k}(a_1; b_1, ..., b_k)a_2 = \\ &\textstyle\sum_{p=1,...,k-1} E_{1,p}(a_1; b_1, ..., b_p) \cdot E_{1,k-p}(a_2; b_{p+1}, ..., b_k); \end{aligned} \tag{10}$$

$$\begin{aligned} &E_{1,n}(E_{1,m}(a; b_1, ..., b_m); c_1, ..., c_n) = \\ &\textstyle\sum E_{1,n-\sum l_i+m}(a; c_1, ..., c_{k_1}, E_{1,l_1}(b_1; c_{k_1+1}, ..., c_{k_1+l_1}), c_{k_1+l_1+1}, ..., c_{k_m}, \\ &E_{1,l_m}(b_m; c_{k_m+1}, ..., c_{k_m+l_m}), c_{k_m+l_m+1}, ..., c_n). \end{aligned} \tag{11}$$

Let us analyze these conditions in low dimensions.

For the operation $E_{1,1}$ the condition (9) gives

$$dE_{1,1}(a;b) + E_{1,1}(da;b) + E_{1,1}(a;db) = a \cdot b + b \cdot a, \tag{12}$$

i.e. the operation $E_{1,1}$ is sort of $\smile_1$ product, which measures the noncommutativity of A. Bellow we use the notation $E_{1,1} = \smile_1$.

The condition (10) gives

$$(a \cdot b) \smile_1 c + a \cdot (b \smile_1 c) + (a \smile_1 c) \cdot b = 0, \tag{13}$$

that is our $E_{1,1} = \smile_1$ satisfies the *Hirsch formula*.

The condition (9) gives

$$\begin{gathered} a \smile_1 (b \cdot c) + b \cdot (a \smile_1 c) + (a \smile_1 b) \cdot c = \\ dE_{1,2}(a;b,c) + E_{1,2}(da;b,c) + E_{1,2}(a;db,c) + E_{1,2}(a;b,dc), \end{gathered} \tag{14}$$

so the "left Hirsch formula" is satisfied just up to chain homotopy and the suitable homotopy is operation $E_{1,2}$, so this operation measures the lack of "left Hirsch formula".

Besides, the condition (11) gives

$$(a \smile_1 b) \smile_1 c - a \smile_1 (b \smile_1 c) = E_{1,2}(a;b,c) + E_{1,2}(a;c,b), \tag{15}$$

so this $\smile_1$ is not strictly associative, but the operation $E_{1,2}$ somehow measures the lack of this associativity too.

7.3 Homotopy G-algebra structure and a multiplication in the bar construction

For a homotopy G-algebra $(C, d, \cdot, \{E_{1,k}\})$ the sequence $\{E_{1,k}\}$ defines on the bar construction BC of a dg algebra $(A, d, \cdot)$ a multiplication which turns BC into a dg bialgebra. In fact this means that a homotopy G-algebra is a $B(\infty)$-algebra in the sense of [11].

The sequence of operations $\{E_{1,k}\}$ defines a homomorphism $E : BC \otimes BC \to C$ by $E([\,]\otimes[a]) = E([a]\otimes[\,]) = a$, $E([a]\otimes[b_1|...|b_n]) = E_{1,n}(a; b_1, ..., b_n)$ and $E([a_1|...|a_m] \otimes [b_1|...|b_n]) = 0$ if $m > 1$.

Since the bar construction BA is a cofree coalgebra, a homomorphism E coextends to a graded coalgebra map $\mu_E : BC \otimes BC \to BC$.

Then the conditions (9) and (10) are equivalent to the condition

$$dE + E(d_{BC} \otimes id + id \otimes d_{BC}) + E \smile E = 0,$$

that is E is a twisting cochain, and this is equivalent to μ_E being a chain map. Besides, the condition (11) is equivalent to μ_E being associative. Finally we have

Proposition 21 *For a homotopy G-algebra $(C, d, \cdot, \{E_{1,k}\})$ the bar construction BC is a dg bialgebra with respect to the standard coproduct $\nabla_B : BC \to BC \otimes BC$ and the multiplication $\mu_E : BC \otimes BC \to BC$.*

Here are the formulas for μ_E in low dimensions:

$$\begin{gathered}
\mu_E([a] \otimes [b]) = [a, b] + [b, a] + [a \smile_1 b], \\
\mu_E([a_1, a_2] \otimes [b]) = [a_1, a_2, b] + [a_1, b, a_2] + [b, a_1, a_2] + \\
[a_1] \otimes [a_2 \smile_1 b] + [a_1 \smile b] \otimes [a_2], \\
\mu_E([a] \otimes [b_1, b_2]) = [a, b_1, b_2] + [b_1, a, b_2] + [b_1, b_2, a] + \\
[b_1] \otimes [a \smile_1 b_2] + [a \smile_1 b_1] \otimes [b_2] + \ [E_{12}(a; b_1, b_2)].
\end{gathered}$$

7.4 hGa Structure on the Cochain Algebra of a Simplicial Complex

The cochain complex of 1-reduced simplicial set $C^*(X)$ is a hGa. The operations $E_{1,k}$ here are dual to cooperations defined by Baues in [2], and the starting operation $E_{1,1}$ is the classical Steenrod's $\smile_1$ product. The description of these operations is much more complicated then the description of $\smile$ and $E_{1,1} = \smile_1$. At the end of these notes we describe them effectively in terms of *surjection operad*.

What is good in this structure?

The homology modules of the bar construction $BC^*(X)$ of the cochain dg algebra coincide with cohomology modules of the loop space $H^*(\Omega X)$. The bar construction $BC^*(X)$ generally is a dg coalgebra, but the hGa structure of $C^*(X)$ produces on $BC^*(X)$ a dg algebra structure, and, in its turn, this structure allows to produce the second bar construction $BBC^*(X)$ which gives cohomology modules of the double loop space $H^*(\Omega^2 X)$.

For the next iterations more operations on $C^*(X)$ are needed, and the operadic language is the best language to describe them.

7.5 Extended homotopy G-algebras

In this section we introduce the notion of *extended homotopy G-algebra* [19]. This is a dg algebra with certain additional structure which defines $\smile_i$-s on the bar construction.

7.5.1 The notion of extended homotopy G-algebra

Definition 17 *An extended homotopy G-algebra we define as an object*

$$(C, d, \cdot, \{E^k_{p,q} : C^{\otimes p} \otimes C^{\otimes q} \to C,\ \ k = 0, 1, ...;\ \ p, q = 1, 2, ...\})$$

such that:

$E^0_{p>1,q} = 0$ *and* $(C, d, \cdot, \{E^0_{1,q}\})$ *is a homotopy G-algebra;*

and

$$\begin{gathered} dE^k_{m,n}(a_1, ..., a_m; b_1, ..., b_n) + \sum_i E^k_{m,n}(a_1, ..., da_i, ..., a_m; b_1, .., b_n)+ \\ \sum_i E^k_{m,n}(a_1, ..., a_m; b_1, ..., db_i, .., b_n)+ \\ \sum_i E^k_{m-1,n}(a_1, ..., a_i \cdot a_{i+1}, ..., a_m; b_1, ..., b_n)+ \\ \sum_i E^k_{m,n-1}(a_1, ..., a_m; b_1, ..., b_i \cdot b_{i+1}, ..., b_n)+ \\ a_1 E^k_{m-1,n}(a_2, ..., a_m; b_1, ..., b_n) + E^k_{m-1,n}(a_1, ..., a_{m-1}; b_1, ..., b_n)a_m \\ +b_1 E^k_{m,n-1}(a_1, ..., a_m; b_2, ..., b_n) + E^k_{m,n-1}(a_1, ..., a_m; b_1, ..., b_{n-1})b_n+ \\ \sum_{i=0}^k \sum_{p,q} T^i E^{k-i}_{p,q}(a_1, ..., a_p; b_1, .., b_q) \cdot E^i_{m-p,n-q}(a_{p+1}, ..., a_m; b_{q+1}, ..., b_n) = \\ E^{k-1}_{m,n}(a_1, ..., a_m; b_1, ..., b_n) + E^{k-1}_{n,m}(b_1, ..., b_n; a_1, ..., a_m), \end{gathered} \tag{16}$$

here $TE^i_{p,q}(x_1, ..., x_p; y_1, ..., y_q) = E^i_{q,p}(y_1, ..., y_q; x_1, ..., x_p)$.

Let us analyze this condition in low dimensions.

For the operation $E^k_{1,1}$ the condition (16) gives

$$dE^k_{1,1}(a; b) + E^k_{1,1}(da; b) + E^k_{1,1}(a; db) = E^{k-1}_{1,1}(a; b) + E^{k-1}_{1,1}(b; a),$$

i.e. the operation $E^k_{1,1}$ is the homotopy which measures the lack of commutativity of $E^{k-1}_{1,1}$. Having in mind that $E^0_{1,1} = \smile_1$ we can say that $E^k_{1,1}$ is sort of $\smile_{k+1}$ product on A. Bellow we use the notation $E^k_{1,1} = \smile_{k+1}$.

Besides the condition (16) also gives

$$\begin{gathered} (a \cdot b) \smile_k c + a \cdot (b \smile_k c) + (a \smile_k c) \cdot b + E^{k-2}_{2,1}(a, b; c) + E^{k-2}_{1,2}(c; a, b) = \\ dE^{k-1}_{2,1}(a, b; c) + E^{k-1}_{2,1}(da, b; c) + E^{k-1}_{2,1}(a, db; c) + E^{k-1}_{2,1}(a, b; dc) \end{gathered} \tag{17}$$

and

$$a \smile_k (b\cdot c) + b\cdot(a \smile_k c) + (a \smile_k b)\cdot c + E^{k-2}_{1,2}(a;b,c) + E^{k-2}_{2,1}(b,c;a) = \\ dE^{k-1}_{1,2}(a;b,c) + E^{k-1}_{1,2}(a;db,c) + E^{k-1}_{1,2}(a;b,dc), \tag{18}$$

these are up to homotopy Hirsch type formulae connecting $\smile_k$ and $\cdot$. We remark here that a homotopy G-algebra structure controls conection between $\cdot$ and $\smile_1$, while the extended homotopy G-algebra structure controls the connections between $\cdot$ and $\smile_k$-s (but not between $\smile_m$ and $\smile_n$ generally).

As we already know a homotopy G-algebra structure defines a multiplication in the bar construction. Bellow we are going to show that an extended homotopy G-algebra structure defines on the bar construction Steenrod $\smile_i$ products. But before we need some preliminary notions.

7.5.2 A structure of extended homotopy G-algebra and Steenrod products in the bar construction

As we already know the part of extended homotopy G-algebra - the sequence of operations $\{E^0_{p,q}\}$ (which in fact is a homotopy G-algebra structure) defines on the bar construction BC a multiplication, turning BC into a dg bialgebra. Here we show that for an extended homotopy G-algebra $(C,d,\cdot,\{E^k_{p,q}\})$ the sequence $\{E^{k>0}_{p,q}\}$ defines in the bar construction BC of a dg algebra $(C,d,\cdot)$ the $\smile_i$-products turning BC into a dg bialgebra with Steenrod products.

Indeed, the sequences of operations $\{E^k_{p,q}\}$ define homomorphisms

$$\{E^k : BC\otimes BC \to C,\ k=0,1,...\}$$

by $E^k([a_1|...|a_m]\otimes[b_1|...|b_n]) = E^k_{m,n}(a_1,...,a_m;b_1,...,b_n)$, which, in its turn defines

$$BC\otimes BC \to BC,\ k=0,1,...\}.$$

The condition (16) guarantees that this multiplication is the correct $\smile_k$-s on BC.

Finally we have

Proposition 22 *For an extended homotopy G-algebra* $(C,d,\cdot,\{E^k_{p,q}\})$ *the*

7.6 Hochschild cochain complex as a hGa

This structure is related to the Deligne's conjecture, which we discuss at the end of these notes.

Let A be an algebra and M be a two sided module on A. The Hochschild cochain complex $C^*(A;M)$ of A with coefficients in M is defined as $C^n(A;M) = Hom(\otimes^n A, M)$ with differential $\delta : C^{n-1}(A;M) \to C^n(A;M)$ given by

$$\delta f(a_1 \otimes ... \otimes a_n) = a_1 \cdot f(a_2 \otimes ... \otimes a_n) +$$
$$\sum_{k=1}^{n-1} f(a_1 \otimes ... \otimes a_{k-1} \otimes a_k \cdot a_{k+1} \otimes .. \otimes a_n) + f(a_1 \otimes ... \otimes a_{n-1}) \cdot a_n.$$

If M is an algebra over A then the Hochschild complex becomes a dg algebra with respect to the $\smile$ product

$$f \smile g(a_1 \otimes ... \otimes a_{n+m}) = f(a_1 \otimes ... \otimes a_n) \cdot g(a_{n+1} \otimes ... \otimes a_{n+m}).$$

We focus on the case $M = A$. In this case on the Hochschild cochain complex $C^*(A;A)$ there exists a structure of homotopy G-algebra. The operations forming this structure were constructed in [15], [11], [9]. Below we describe this structure.

In [8] Gerstenhaber has defined in the Hochschild complex $C^*(A,A)$ a product $f \circ g$ given by

$$f \circ g(a_1 \otimes ... \otimes a_{n+m-1}) =$$
$$\sum_{k=0}^{n-1} f(a_1 \otimes ... \otimes a_k \otimes g(a_{k+1} \otimes .. \otimes a_{k+m}) \otimes a_{k+m+1} \otimes ... \otimes a_{n+m-1}).$$

The Gerstenhaber's product has the following properties:

$$\delta(f \circ g) = \delta f \circ g + f \circ \delta g + f \smile g - g \smile f,$$

and

$$(f \smile g) \circ h = f \smile (g \circ h) + (f \circ h) \smile g,$$

this means, that the product $f \circ g$ has the properties of $\smile_1$ product: if we use the notation $f \circ g = f \smile_1 g$, then the first condition turns the Steenrod formula

$$\delta(f \smile_1 g) = \delta f \smile_1 g + f \smile_1 \delta g + f \smile g - g \smile f$$

and the second turns to the left Hirsch formula

$$(f \smile g) \smile_1 h = f \smile (g \smile_1 h) + (f \smile_1 h) \smile g.$$

As for right Hirsch formula, in [8] there is defined $\smile_1$ product of a cochain $f \in C^p(A;A)$ and a couple of cochains $g \in C^q(A;A),\ h \in C^r(A;A)$:

$$\begin{gathered}(f \smile_1 (g,h))(a_1 \otimes ... \otimes a_{p+q+r-2}) = \\ \sum_{k,l} f(a_1 \otimes ... \otimes a_k \otimes g(a_{k+1} \otimes ... \otimes a_{k+q}) \otimes a_{k+m+1} \otimes ... \\ \otimes a_l \otimes h(a_{l+1} \otimes ... \otimes a_{l+r}) \otimes a_{l+r+1} \otimes ... \otimes a_{p+q+r-2}).\end{gathered}$$

The straightforward verification shows, that the $\smile_1$ product in $C^p(A;A)$ satisfies the right Hirsch formula up to homotopy and the appropriate homotopy is $f \smile_1 (g,h)$, i.e. the following condition is satisfied

$$\begin{gathered}\delta(f \smile_1 (g,h)) + \delta f \smile_1 (g,h) + f \smile_1 (\delta g,h) + f \smile_1 (g,\delta h) = \\ f \smile_1 (g \smile h) + g \smile (f \smile_1 h) + (f \smile_1 g) \smile h.\end{gathered}$$

Let us mention also the following property of the introduced product: the product $f \smile_1 (g,h)$ measures the nonassociativity of $\smile_1$ product:

$$f \smile_1 (g \smile_1 h) - (f \smile_1 g) \smile_1 h = f \smile_1 (g,h) + f \smile_1 (h,g). \qquad (19)$$

Remark 1 *In [8], see also [32], in the desuspension of Hochschild complex* $s^{-1}C^*(A;A)$ *a dg Lie algebra structure was introduced. Actually the Lie bracket* $[f,g]$ *is the commutator of* $\smile_1$ *product:* $[f,g] = f \smile_1 g - g \smile_1 f$. *Although the* $\smile_1$ *product is not associative, the condition (19) allows to check, that the Jacobi identity is satisfied.*

In [15] (see also [11], [9]) there is defined the brace operation $f\{g_1,...,g_i\}$ of a Hochschild cochain $f \in C^n(A,A)$ and a sequence of cochains $g_k \in C^{n_k}(A,A)$:

$$\begin{gathered}f\{g_1,...,g_i\}(a_1 \otimes ... \otimes a_{n+n_1+...+n_i-i}) = \\ \sum_{k_1,...,k_i} f(a_1 \otimes ... \otimes a_{k_1} \otimes g_1(a_{k_1+1} \otimes ... \otimes a_{k_1+n_1}) \otimes a_{k_1+n_1+1} \otimes ... \otimes \\ a_{k_2} \otimes g_2(a_{k_2+1} \otimes ... \otimes a_{k_2+n_2}) \otimes a_{k_2+n_2+1} \otimes ... \\ a_{k_i} \otimes g_i(a_{k_i+1} \otimes ... \otimes a_{k_i+n_i}) \otimes a_{k_i+n_i+1} \otimes ... \otimes a_{n+n_1+...+n_i-i}).\end{gathered} \qquad (20)$$

The straightforward verification shows that the collection $\{E_{1,k}\}$ given by

$$E_{1,k}(f;g_1,...,g_k) = f\{g_1,...,g_k\}$$

satisfies the needed conditions, thus it forms on the Hochschild complex $C^*(A;A)$ is a structure of homotopy G-algebra.

8 Differential graded Operads

To handle the above described huge sets of cochain operations, the universal notion of operad is was introduced. There is wide literatire about the subject, see [22], [30], [24], [25], [4], etc.

The notion of dg operad can be considered as a fahr developed generalization of the classical notion of an algebra. Here step by step we introduce a sequence of notion which will end by the notion of dg operad.

8.0.1 Algebra

Algebra is a vector space A together with a multiplication $a \cdot b \in A$ which satisfies some natural properties (unit, distributivity and associativity).

Generic example. Let V be a vector space. Then $End(V)$ is an algebra with the following structure: For $f, g \in End(V)$

$$(f+g)(v) = f(v) + g(v),\ \ f \cdot g(v) = f(g(v)).$$

An algebra A *acts* on V, in other words V is an A-module, if there exists a map of algebras

$$A \to End(V).$$

8.0.2 Graded Algebra

Graded algebra is a collection of vector spaces

$$...,\ \ A_0,\ \ A_1,\ \ A_2,\ \ ...\ \ , A_n,\ \ ...$$

with a multiplication $a_m \cdot b_n \in A_{m+n}$.

In particular A_0 is an algebra.

Generic example. Let $V_* = \{..., V_n, V_{n+1}, ...\}$ be a graded vector space. Then

$$End(V_*) = \{..., End^n(V_*), End^{n+1}(V_*), ...\}$$

where $End^n(V) = \{f : V_* \to V_{*+n}\}$ consists of endomorphisms of V_* of degree n, is a graded algebra with the following structure: For $f, g \in End(V_*)$

$$(f+g)(v) = f(v) + g(v),\ \ f \cdot g(v) = f(g(v)).$$

A graded algebra A *acts* on V, in other words V is an A-module, if there exists a map of graded algebras

$$A \to End(V_*).$$

8.0.3 Differential Graded Algebra

Differential graded algebra is a graded algebra with differentials

$$... \stackrel{d}{\leftarrow} A_0 \stackrel{d}{\leftarrow} A_1 \stackrel{d}{\leftarrow} A_3 \stackrel{d}{\leftarrow} ... \stackrel{d}{\leftarrow} A_n \stackrel{d}{\leftarrow} ...$$

with a multiplication $a_m \cdot b_n \in A_{m+n}$ and differential $d : A_n \to A_{n-1}$ such that

$$d(a_m \cdot b_n) = d(a_m) \cdot b_n + (-1)^n a_m \cdot d(b_n).$$

In particular A_0 is an algebra.

Generic example. Let $(V_*, d) = \{... \leftarrow V_n \leftarrow V_{n+1} \leftarrow ...\}$ be a dg module (chain complex). Then $End(V_*)$ with above described graded algebra structure and the differential

$$D : End^n(V_*) \to End^{n-1}(V_*)$$

defined by

$$D(f) = df - (-1)^n fd$$

is a dg algebra.

A dg algebra A *acts* on V, in other words V is an A-module, if there exists a map of dg algebras

$$A \to End(V_*).$$

8.0.4 Differential Graded Operad

Definition 18 *Differential graded operad A is a sequence of chain complexes*

$$\begin{array}{l}
... \stackrel{d}{\leftarrow} A(1)_0 \stackrel{d}{\leftarrow} A(1)_1 \stackrel{d}{\leftarrow} A(1)_3 \stackrel{d}{\leftarrow} ... \stackrel{d}{\leftarrow} A(1)_n \stackrel{d}{\leftarrow} ... \\
.... \stackrel{d}{\leftarrow} A(2)_0 \stackrel{d}{\leftarrow} A(2)_1 \stackrel{d}{\leftarrow} A(2)_3 \stackrel{d}{\leftarrow} ... \stackrel{d}{\leftarrow} A(2)_n \stackrel{d}{\leftarrow} ... \\
... \stackrel{d}{\leftarrow} A(3)_0 \stackrel{d}{\leftarrow} A(3)_1 \stackrel{d}{\leftarrow} A(3)_3 \stackrel{d}{\leftarrow} ... \stackrel{d}{\leftarrow} A(3)_n \stackrel{d}{\leftarrow} ... \\
... \quad ... \quad ... \\
... \stackrel{d}{\leftarrow} A(k)_0 \stackrel{d}{\leftarrow} A(k)_1 \stackrel{d}{\leftarrow} A(k)_3 \stackrel{d}{\leftarrow} ... \stackrel{d}{\leftarrow} A(k)_n \stackrel{d}{\leftarrow} ... \\
... \quad ... \quad ...
\end{array}$$

equipped with the following structure:

(i) Each $A(n)_*$ is a Σ_n-module, that is the symmetric group Σ_n acts on the chain complex $A(n)_*$.

(ii) There is defined a (partial) composition product

$$a(k)_m \circ_i b(t)_n \in A(k+t-1)_{m+n}, \; i = 1, 2, ..., k$$

so that the following conditions are satisfied:
Associativity. *For $f \in A(n)_*, \; g \in A(p)_*, \; h \in A(q)_*$*

$$\begin{array}{l}
f \circ_i (g \circ_j h) = (f \circ_i g) \circ_{i+j-1} h, \quad 1 \le i \le n, \; 1 \le j \le p; \\
(f \circ_j h) \circ_i g = (f \circ_i g) \circ_{p+j-1} h, \quad 1 \le i < j \le n.
\end{array}$$

Compatibility with differential. *The composition product*

$$\circ_i : A(k)_* \otimes A(t)_* \to A(k+t)_*$$

is a chain map

$$d(f \circ_i g) = d(f) \circ_i g + (-1)^n f \circ_i d(g).$$

Compatibility with symmetric group action. *See bellow.*
Unit. *There exists $e \in A(1)_0$ such that*

$$a \circ_i e = e \circ_1 a = a.$$

Homology of operad $A(*)_*$ is an operad HA with $HA(n)_k = H_k(A(n)_*, d)$, trivial differential and the induced composition product and Σ_* action.

Remark. The composition product

$$\circ_i : A(m) \otimes A(n) \to A(m+n-1),\ \ i = 1, 2, ..., m$$

defines the *circle* product

$$\circ : A(m) \otimes (A(k_1)) \otimes ... \otimes A(k_m)) \to A(k_1 + ... + k_m)$$

by

$$f \circ (g_1, ..., g_m) = (...(f \circ_1 g_1) \circ_2 g_2) \circ_3 ...) \circ_m g_m).$$

Remark. Here we explain how the composition product is compatible with symmetric group action.

For $f \in A(m)_*,\ g \in A(n)_*,\ \sigma \in \Sigma_m,\ \tau \in \Sigma_n$

$$(\sigma \cdot f) \circ_i g = (\sigma \circ_i e_{\Sigma_n)} \cdot (f \circ_{\sigma^{-1}(i)} g);$$
$$f \circ_i (\tau \cdot g) = (e_{\Sigma_m)} \circ_i \tau \cdot (f \circ_i g);$$

where $\sigma \circ_i \tau \in \Sigma_{m+n-1}$ is the permutation obtained from $\sigma = (\sigma(1), ..., \sigma(m))$ by substituting of $\tau = (\tau(1), ..., \tau(n))$ at the place where the number i occurs, here is the procedure. Suppose i occurs at the place t, i.e. $i = \sigma(t)$, then:

We leave unchanged all $\sigma(k)$-s which are less then i,

furthermore, we replace $i = \sigma(t)$ by the sequence $(v(1)+i-1, ..., v(n)+i-1)$,

and finally increase all $u(k)$-s which are more then i by $n-1$.

For example

$$(3, 1, \mathbf{2}) \circ_2 (1, 3, 2) = (5, 1, \mathbf{2}, \mathbf{4}, \mathbf{3}).$$

Remark. In particular $A(1)_0$ is an algebra and the first row $A(1)_*$ is a dg algebra. So the notion of dg operad generalizes that of dg algebra.

Remark. There is a notion of non-Σ_* operad, where the action of symmetric group is dropped.

Definition 19 *A morphism of operads $A \to A'$ is a collection of chain maps*

$$f(n) : A(n)_* \to A'(n)_*$$

which preserves the symmetric group action and the composition product.

Generic example. Let again (V, d) be a dg module(chain complex). The *endomorphism operad* of V is defined as follows. Let $A(n)_k = Hom^k(V^{\otimes n}, V) = \{f : (V^{\otimes n})_* \to V_{*+k}\}$. Then $A(n)_* = Hom^*(V^{\otimes n}, V)$ is a chain complex with differential

$$Df(a_1 \otimes ... \otimes a_n) = df(a_1 \otimes ... \otimes a_n) \pm \sum_i f(a_1 \otimes ... \otimes d(a_i) \otimes ... \otimes a_n).$$

We claim that

$$\begin{array}{lllllllll}
... \leftarrow & Hom^0(V,V) & \leftarrow & Hom^1(V,V) & \leftarrow & Hom^2(V,V) & \leftarrow & ... & , \\
... \leftarrow & Hom^0(V^{\otimes 2},V) & \leftarrow & Hom^1(V^{\otimes 2},V) & \leftarrow & Hom^2(V^{\otimes 2},V) & \leftarrow & ... & , \\
... \leftarrow & Hom^0(V^{\otimes 3},V) & \leftarrow & Hom^1(V^{\otimes 3},V) & \leftarrow & Hom^2(V^{\otimes 3},V) & \leftarrow & ... & , \\
... & ... & & ... & & & & &
\end{array}$$

is an dg operad with clear symmetric group action and the following *composition product*: for

$$f \in Hom^p(V^{\otimes m}, V),\ g \in Hom^q(V^{\otimes n}, V)$$

the product $f \circ_k g \in Hom^{p+q}(V^{\otimes(m+n-1)}, V),\ k = 1, 2, ..., m$ is defined by

$$\begin{array}{l}
f \circ_k g(v_1 \otimes ... \otimes v_{m+n-1}) = \\
f(v_1 \otimes ... \otimes v_{k-1} \otimes g(v_k \otimes ... \otimes v_{k+n-1}) \otimes ... \otimes v_{m+n-1}).
\end{array}$$

The circle product is the substitution

$$\begin{array}{l}
f \circ (g_1, ..., g_m)(a_1 \otimes ... \otimes a_{k_1+...+k_m}) = \\
\sum f(g_1(a_1 \otimes ... \otimes a_{k_1}) \otimes ... \otimes g_{k_m}(a_{k_1+...+k_{m-1}} \otimes ... \otimes a_{k_1+...+k_m})).
\end{array}$$

The defining conditions of an operad mimic the properties of the endomorphism opearad.

The endomorphism operad inspires the following terminology: An element $a \in A(n)_k$ has the *degree* k and the *arity* n.

8.0.5 Interpretation of Some Cochain Operations in Operadic terms

If V is a dg algebra then the multiplication $\smile: V \otimes V \to V$ can be considered as an element $\smile \in Hom^0(V^{\otimes 2}, V)$. The condition

$$d(a \smile b) = da \smile b \pm a \smile db$$

means that $\smile \in Hom^0(V,V)$ is a cycle in endomorphism operad.

If in addition V possesses a $\smile_1$ product, then this operation can be considered as an element $\smile_1 \in Hom^{-1}(V^{\otimes 2},V)$. The property

$$d(a\smile_1 b) = da\smile_1 b \pm a\smile_1 db \pm a\smile b \pm b\smile a$$

in terms of endomorphism operad means

$$D(\smile_1) = \smile \pm T\cdot \smile,$$

where T is nonidentical element is the symmetric group Σ_2.

Furthermore, if V possesses the operation E_{12}, see (7), then this operation can be considered as an element $E_{12} \in Hom^{-2}(V^{\otimes 3},V)$. The condition

$$\begin{array}{l} a\smile_1 (b\smile c) \pm b\smile (a\smile_1 c) \pm (a\smile_1 b)\smile c = \\ dE_{1,2}(a;b,c) + E_{1,2}(da;b,c) + E_{1,2}(a;db,c) + E_{1,2}(a;b,dc). \end{array}$$

in terms of endomorphism operad means

$$\smile_1 \circ_2 \smile \pm (T\times id)\cdot \smile \circ_2 \smile_1 \pm \smile \circ_1 \smile_1 = DE_{12}.$$

8.1 Algebras over Operads

Definition 20 *A dg operad $A(*)_*$* acts *on (V,d), in other words (V,d) is an algebra over $A(*)_*$, if there exists a map of dg operads*

$$A(*)_* \to Hom^*(V^{\otimes *},V).$$

Clearly V is an algebra over the endomorphism operad $Hom^*(V^{\otimes *},V)$.

8.2 Commutative Operad

Commutative operad $Comm$ is defined as a sequence of chain complexes

$$\begin{array}{l}
... \leftarrow 0 \stackrel{d}{\leftarrow} Comm(1)_0 \stackrel{d}{\leftarrow} 0 \stackrel{d}{\leftarrow} 0 \stackrel{d}{\leftarrow} ... \stackrel{d}{\leftarrow} 0 \stackrel{d}{\leftarrow} ... \\
... \leftarrow 0 \stackrel{d}{\leftarrow} Comm(2)_0 \stackrel{d}{\leftarrow} 0 \stackrel{d}{\leftarrow} 0 \stackrel{d}{\leftarrow} ... \stackrel{d}{\leftarrow} 0 \stackrel{d}{\leftarrow} ... \\
... \leftarrow 0 \stackrel{d}{\leftarrow} Comm(3)_0 \stackrel{d}{\leftarrow} 0 \stackrel{d}{\leftarrow} 0 \stackrel{d}{\leftarrow} ... \stackrel{d}{\leftarrow} 0 \stackrel{d}{\leftarrow} ... \\
... \quad ... \quad ... \\
... \leftarrow 0 \stackrel{d}{\leftarrow} Comm(n)_0 \stackrel{d}{\leftarrow} 0 \stackrel{d}{\leftarrow} 0 \stackrel{d}{\leftarrow} ... \stackrel{d}{\leftarrow} 0 \stackrel{d}{\leftarrow} ... \\
... \quad ... \quad ...
\end{array}$$

where each $Comm(n)_0$ is the ground ring R with generator a_n

$$Comm(n)_0 = R \cdot a_n.$$

The differential is clearly trivial, as well as symmetric group actions, and the composition product given by $a_p \circ_i a_q = a_{p+q-1}$.

Algebras over $Comm$ are commutative dg algebras: If $Comm$ acts on V, i.e. an operadic map

$$f : Comm(n) \to Hom^*(V^{\otimes *}, V)$$

is given, then V is a graded commutative and associative algebra with multiplication $f(a_2) = \mu : V \otimes V \to V$:

$$\begin{array}{lll} a_1 & \longrightarrow & id \in Hom(V,V) \\ a_2 & \longrightarrow & \mu \in Hom(V^{\otimes 2}, V) \\ a_3 = a_2 \circ_1 a_2 = a_2 \circ_2 a_2 & \longrightarrow & \mu(\mu \otimes id) = \mu(id \otimes \mu) \in Hom(V^{\otimes 2}, V) \\ \ldots & & \end{array}$$

and $a_2 = T \cdot a_2$ implies $\mu = \mu T$, where $T \in \Sigma_2$ is nonidentical permutation.

Homology of the operad $Comm$ is the operad $Comm$ itself.

8.3 Associative Operad

Associative operad Ass is defined as a sequence of chain complexes

$$\begin{array}{l} \ldots \leftarrow 0 \xleftarrow{d} Ass(1)_0 \xleftarrow{d} 0 \xleftarrow{d} 0 \xleftarrow{d} \ldots \xleftarrow{d} 0 \xleftarrow{d} \ldots \\ \ldots \leftarrow 0 \xleftarrow{d} Ass(2)_0 \xleftarrow{d} 0 \xleftarrow{d} 0 \xleftarrow{d} \ldots \xleftarrow{d} 0 \xleftarrow{d} \ldots \\ \ldots \leftarrow 0 \xleftarrow{d} Ass(3)_0 \xleftarrow{d} 0 \xleftarrow{d} 0 \xleftarrow{d} \ldots \xleftarrow{d} 0 \xleftarrow{d} \ldots \\ \ldots \quad \ldots \quad \ldots \\ \ldots \leftarrow 0 \xleftarrow{d} Ass(n)_0 \xleftarrow{d} 0 \xleftarrow{d} 0 \xleftarrow{d} \ldots \xleftarrow{d} 0 \xleftarrow{d} \ldots \\ \ldots \quad \ldots \quad \ldots \end{array}$$

where each $Ass(n)_0$ is the free Σ_n-module with one generator a_n, i.e

$$Ass(n) = \sum_{\sigma \in \Sigma_n} R \cdot \sigma a_n,$$

that is a free R-module with $n!$ generators, with trivial differential and the composition product given by $a_p \circ_i a_q = a_{p+q-1}$.

Algebras over Ass are (associative) dg algebras: If Ass acts on V, i.e. an operadic map

$$f : Ass(n) \to Hom^*(V^{\otimes *}, V)\}$$

is given, then V is a graded associative algebra with multiplication $f(a_2) = \mu : V \otimes V \to V$:

$$\begin{array}{lcl} a_1 & \longrightarrow & id \in Hom(V,V) \\ a_2 & \longrightarrow & \mu \in Hom(V^{\otimes 2}, V) \\ a_3 = a_2 \circ_1 a_2 = a_2 \circ_2 a_2 & \longrightarrow & \mu(\mu \otimes id) = \mu(id \otimes \mu) \in Hom(V^{\otimes 2}, V) \\ ... & & \end{array}$$

Homology of the operad $Comm$ is the operad $Comm$ itself.

There is an obvious operadic map $Ass \to Comm$.

8.4 A_∞-operad

May: An A_∞ operad is a non-Σ operad with each $A(n)$ contractible.

$A(\infty)$-operad is the free acyclic non-Σ_* operad generated by elements $m_k \in A(k)_{k-2},\ k = 2, 3, 4, ...$ with differential

$$dm_k = \sum_{p+q=k+1} \sum_{j=1}^{p} \pm m_p \circ_j m_q.$$

Here is the operad $A(\infty)$ in low dimensions

$$\begin{array}{lcccccccc}
A(1) & = & (<1> & \leftarrow & 0 & \leftarrow & 0 & \leftarrow ...) \\
A(2) & = & (< m_2 > & \leftarrow & 0 & \leftarrow & 0 & \leftarrow ...) \\
A(3) & = & (< m_2 \circ_1 m_2, & \leftarrow & < m_3 > & \leftarrow & 0 & \leftarrow ...) \\
 & & m_2 \circ_2 m_2 > & & & & & \\
A(4) & = & (<\ (m_2 \circ_1 m_2) \circ_1 m_2, & \leftarrow & < m_2 \circ_1 m_3, & \leftarrow & < m_4 > & \leftarrow ...) \\
 & & (m_2 \circ_1 m_2) \circ_2 m_2, & & m_2 \circ_2 m_3, & & & \\
 & & (m_2 \circ_2 m_2) \circ_2 m_2, & & m_3 \circ_1 m_2, & & & \\
 & & (m_2 \circ_2 m_2) \circ_3 m_2, & & m_3 \circ_2 m_2, & & & \\
 & & (m_2 \circ_2 m_2) \circ_1 m_2 > & & m_3 \circ_3 m_2 > & & & \\
...
\end{array}$$

$$dm_3 = m_2 \circ_1 m_2 + m_2 \circ_2 m_2,$$
$$dm_4 = m_3 \circ_1 m_2 + m_3 \circ_2 m_2 + m_3 \circ_3 m_2 + m_2 \circ_1 m_3 + m_2 \circ_2 m_3$$
$$...$$

8.4.1 A_∞-algebra structure in Homology of a dg Algebra

$\mathcal{A}(\infty)$ is a *free resolution* of *Ass*: the projection $\mathcal{A}(\infty) \to Ass$ is weak equivalence of operads.

Theorem 6 *Let a chain complex C be an algebra over an operad $\mathcal{P}$ and $\mathcal{R} \to \mathcal{P}$ be a free resolution of $\mathcal{P}$. Then $H(C)$ is an algebra over $\mathcal{R}$.*

Corollary 1 *If C is an algebra over the associative operad Ass then the homology $H(C)$ is an algebra over the $A(\infty)$ operad $A(\infty)$.*

8.5 $E(\infty)$ operad

An E_∞ operad is an operad with each $A(n)$ contractible and σ-free.

8.5.1 The Surjection Operad

The surjection operad $\mathcal{X}$ is one of the E_∞ operads see [25], [4]. All cochain operations described earlier in these notes, such as $\smile$, $\smile_i$ products, operations $E_{1,k}$ which for homotopy G-algebra structure, have nice description in $\mathcal{X}$.

The module $\mathcal{X}(n)_d$ is generated by all maps

$$u : (1, ..., n + d) \to (1, ..., n)$$

but

(i) all non-surjections represent zero element;

(ii) degenerate maps, i.e.such that $u(i) = u(i+1)$ for some i also represent zero element.

Any such map u is represented by a sequence $(u(1), ..., u(n+d))$. So non-surjections and sequences with repetitions represent zero elements in $\mathcal{X}(n)_d$.

The following structure defines the surjection operad $\mathcal{X}$.

The symmetric group Σ_n acts on $\mathcal{X}(n)_d$ by

$$\sigma \cdot u = (\sigma(u(1)), ..., \sigma(u(n + d))).$$

The differential $d : \mathcal{X}(n)_d \to \mathcal{X}(n)_{d-1}$ is the sum

$$d(u(1), ..., u(n+d)) = \sum \pm(u(1), ..., u(i-1), \widehat{u(i)}, u(i+1), ..., u(n+d)).$$

The composition product $u \circ_k v$ for $u \in \mathcal{X}(n)_d$ and $v \in \mathcal{X}(s)_e$ is defined as follows.

(i) Suppose first that in $(u(1), ..., u(n+d))$ there is only one occurrence of the number k and $k = u(m)$. Then we must live unchanged all $u(i)$-s which are less then k, furthermore, replace $u(m)$ by the sequence $(v(1)+k-1, ..., v(s)+k-1)$, and finally increase all $u(i)$-s which are more then k by $s-1$.

(ii) If there are in $(u(1), ..., u(n+d))$ two occurrences of k, say $u(p)$ and $u(q)$, $q > p$, then we split $(v(1), ..., v(s+e))$ into two parts

$$(v(1), ..., v(j)), \quad (v(j), ..., v(s+e)),$$

then produce two sequences

$$(v(1)+k-1, ..., v(j)+k-1), \quad (v(j)+k-1, ..., v(s+e)+k-1),$$

and substitute them instead $u(p)$ and $u(q)$ respectively. Furthermore, we keep all $u(i)$-s which are less then k unchanged an all $u(i)$-s which are more then k increase by $s-1$. The product $u \circ_k v$ then is the sum of all surjections obtained this way.

...

(r) If there are r occurrences of k then we split v into r parts and do the similar substitutions. The product $u \circ_k v$ then is the sum of all surjections obtained this way.

For example

$$(1,2,1,3) \circ_1 (1,2,1) = \pm(1,3,1,2,1,4) \pm (1,2,3,2,1,4) \pm (1,2,1,3,1,4).$$

Here are some generators of $\mathcal{X}$ in low dimensions:

$$\begin{array}{lcccccccl}
\mathcal{X}(1) & = & (1) & \leftarrow & 0 & \leftarrow & 0 & \leftarrow ...) \\
\\
\mathcal{X}(2) & = & (1,2) & \leftarrow & (1,2,1) & \leftarrow & (1,2,1,2) & \leftarrow ...) \\
 & & (2,1) & & (2,1,2) & & (2,1,2,1) & \\
\\
\mathcal{X}(3) & = & (1,2,3) & \leftarrow & (1,2,3,1) & \leftarrow & (1,2,1,3,1) & \leftarrow ...) \\
 & & (1,3,2) & & (1,2,1,3) & & (1,2,3,2,1) & \\
 & & (2,1,3) & & (1,2,3,2) & & (1,2,3,1,2) & \\
 & & ... & & ... & & ... &
\end{array}$$

and an example of differential

$$\begin{aligned} d(1,2,1) = \ & (\hat{1},2,1) + (1,\hat{2},1) + (1,2,\hat{1}) = \\ & (1,2) + (1,1) + (2,1) = (1,2) + (2,1). \end{aligned}$$

8.5.2 Contraction of $\mathcal{X}$

The surjection operad is an E_∞ operad: It is Σ-free contractible operad. The last means that each chain complex

$$\mathcal{X}(n)_0 \overset{d}{\leftarrow} \mathcal{X}(n)_0 \overset{d}{\leftarrow} \mathcal{X}(n)_0 \overset{d}{\leftarrow} ...$$

is contractible, see [26].

There are of course various contraction homotopies. We present here two simplest ones.

The contraction homotopy

$$s : \mathcal{X}(n)_k \overset{d}{\rightarrow} \mathcal{X}(n)_{k+1}$$

adds the number 1 to the beginning of a surjection, so it is given by

$$s(u(1), ..., u(n+k)) = (1, u(1), ..., u(n+k)).$$

The other one is

$$S : \mathcal{X}(n)_k \overset{d}{\rightarrow} \mathcal{X}(n)_{k+1}$$

adds the number 1 to the end of a surjection, so it is given by

$$S(u(1), ..., u(n+k)) = (u(1), ..., u(n+k), 1).$$

Both of them (almost) satisfy

$$ds + sd = id, \qquad dS + Sd = id.$$

Remark. The condition $ds+sd = id$ fails for surjections of type $(1, a, b, c, ...)$ which start with 1 and this is the unique occurrence of 1. Similarly, the condition $ds + sd = id$ fails for surjections of type $(..., x, y, z, 1)$ which end with 1 and this is the unique occurrence of 1. So for *any* surjection we can use either s or S.

8.6 The Action of the Surjection Operad on the Chain Complex of a Simplicial Complex

Let X be a simplicial complex with ordered vertexes. Here we explain how the surjection operad $\mathcal{X}$ acts on the chain complex $C_*(X)$.

Roughly speaking the procedure is following. Let us take an element $u = (u(1), ..., u(n)) \in \mathcal{X}(n)_d$. This element must define an n-ary chain (co)operation of degree d

$$C_*(X) \to C_*(X) \otimes ... \otimes C_*(X)$$

as follows. Take a simplex $\sigma = (v_0, ..., v_m) \in C_m(X)$, and let us first cut this simplex into n parts

$$(v_0, ..., v_{i_1}), \quad (v_{i_1}, ..., v_{i_2}), \quad ..., \quad (v_{i_{n-2}}, ..., v_{i_{n-1}}), \quad (v_{i_{n-1}}, ..., v_{i_n} = v_m),$$

here $0 \le i_1 \le ... \le i_n \le m$. Next we mark one by one each of these intervals by numbers $u(1), ..., u(n)$. Then we collect all the intervals marked by the number 1 together and this will be the first tensor factor. Similarly, we gather all the intervals marked by 2 and this will be the second tensor factor, etc. Acting this way we obtain en element of $C_*(X)^{\otimes n}$. Finally $u(v_0, ..., v_m)$ is the sum of all such elements where the summation is taken by all cuttings of σ into n parts.

Bellow we explain this for some particular elements of $\mathcal{X}$.

Assigning to u the operations

$$C^*(X) \otimes ... \otimes C^*(X) \to C^*(X)$$

dual to the cooperation constructed above we obtain an operadic map

$$\mathcal{X} \to Hom^*(C^*(X)^{\otimes *}, C^*(X)),$$

thus $C^*(X)$ is an algebra over the surjection operad $\mathcal{X}$.

8.6.1 $\smile$-Product in $\mathcal{X}$

Let us start with the element $u = (1,2) \in \mathcal{X}(n)_d = \mathcal{X}(2)_0$. By the above algorithm this element defines the $n = 2$-ary operation of degree $d = 0$

$$(1,2) : C_*(X) \to C_*(X) \otimes C_*(X)$$

as follows.

Take a simplex $\sigma = (v_0, ..., v_m)$. First we divide this simplex in $n = 2$ parts

$$(v_0, ..., v_k), \qquad (v_k, ..., v_m).$$

Next we mark each interval by elements of $u = (1,2)$, so the first interval $(v_0, ..., v_k)$ is marked by 1 and the second interval $(v_k, ..., v_m)$ by 2. So we get

$$(1,2)(v_0, ..., v_m) = \sum_k (v_0, ..., v_k) \otimes (v_k, ..., v_m).$$

This is nothing else than the Alexander-Whitney diagonal, dual to the $\smile$-product. So this important cooperation is represented by the element $(1,2) \in \mathcal{X}(2)_0$.

8.6.2 $\smile_1$-Product in $\mathcal{X}$

The $\smile_1$ product is a tool which measures the deviation from commutativity of the $\smile$ product

$$d(a \smile_1 b) + da \smile_1 b + a \smile_1 db = a \smile b - b \smile a.$$

In dual terms

$$D\Delta_1 = \Delta - T \cdot \Delta.$$

In the surjection operad $\mathcal{X}$ the operation $\smile$, or dually the cooperation Δ, is represented by the element $(1,2) \in \mathcal{X}(2)_0$.

Thus the $\smile_1$ must be represented by an element $U \in \mathcal{X}(2)_1$ which satisfies the condition

$$dU = (1,2) - (2,1).$$

Such an element U is easy to construct: just use the contraction homotopy S which puts an extra 1 at the end

$$U = S((1,2) - (2,1)) = (1,2,\mathbf{1}) - (2,1,\mathbf{1}) = (1,2,1).$$

Let us check now which cochain operation $C_*(V) \to C_*(V) \otimes C_*(V)$ determines this element: on a simplex $(v_0, ..., v_m)$ this element acts as

$$(1,2,1)(v_0, ..., v_m) = \sum_{k_1,k_2} (v_0, ..., v_{k_1}, v_{k_2}, ..., v_m) \otimes (v_{k_1}, ..., v_m),$$

and this is exactly the Steenrod's definition of the operation Δ_1, dual to the $\smile_1$ product.

Note that $d(1,2,1) = (1,2) - (2,1)$ means that the Steenrod formula is satisfied already in the operad $\mathcal{X}$.

8.6.3 $\smile_i$-Product in $\mathcal{X}$

Now let us interpret in $\mathcal{X}$ the Steenrod $\smile_i$ products.

The $\smile_2$ product is a binary operation which measures the deviation from commutativity of $\smile_1$ product

$$d(a \smile_2 b) + da \smile_2 b + a \smile_1 db = a \smile_1 b + b \smile_1 a.$$

In dual terms

$$D\Delta_2 = \Delta_1 + T \cdot \Delta_1.$$

And in terms of $\mathcal{X}$

$$dU = (1,2,1) + T \cdot (2,1,2) = (1,2,1) + (2,1,2).$$

Using the contraction s we obtain explicit form for U

$$U = s((1,2,1) + (2,1,2)) = (\mathbf{1}, 1, 2, 1) + (\mathbf{1}, 2, 1, 2,) = 0 + (1,2,1,2),$$

so $\smile_2 = (1,2,1,2) \in \mathcal{X}(2)_2$.

Similar calculation gives $\smile_3 = (1,2,1,2,1) \in \mathcal{X}(2)_3$, $\smile_4 = (1,2,1,2,1,2) \in \mathcal{X}(2)_4$, etc.

8.7 Homotopy G-algebra operations in the surjection operad

8.7.1 Hirsch formula in $\mathcal{X}$

Here we study more detailed the properties of $\smile_1$ product $(1,2,1)$ in $\mathcal{X}$.

The Hirsch formula

$$(a \smile b) \smile_1 c = a \smile (b \smile_1 c) + (a \smile_1 c) \smile b$$

in dual terms looks as

$$(\Delta \otimes id)\Delta_1 + (id \otimes \Delta_1)\Delta + (id \otimes T)(\Delta_1 \otimes id)\Delta = 0.$$

Now replace Δ by $(1,2)$ and Δ_1 by $(1,2,1)$. Then this condition writes

$$(1,2,1) \circ_1 (1,2) + (1,2) \circ_2 (1,2,1) + (id \times T)(1,2) \circ_1 (1,2,1) = 0, \quad (21)$$

which can be easily verified in $\mathcal{X}$. So the Hirsch formula also holds already in the surjection operad.

8.7.2 Operation $E_{1,2}$ in $\mathcal{X}$

We have already mentioned that the Hirsch type combination

$$U(a,b,c) = a \smile_1 (b \smile c) + b \smile (a \smile_1 c) + (a \smile_1 b) \smile c$$

is not generally zero, but it is homological to zero: there exists an operation $E_{1,2}(a;b,c)$, a part of homotopy G-algebra structure, such that

$$\begin{gathered} a \smile_1 (b \smile c) + b \smile (a \smile_1 c) + (a \smile_1 b) \smile c = \\ dE_{1,2}(a;b,c) + E_{1,2}(da;b,c) + E_{1,2}(a;db,c) + E_{1,2}(a;b,dc). \end{gathered} \quad (22)$$

Below we show how one can *discover* the $E_{1,2}$ operation easily using the surjection operad.

In dual terms this combination looks as

$$\begin{aligned} &(id \otimes \Delta)\Delta_1 + (T \otimes id)(id \otimes \Delta_1)\Delta + (\Delta_1 \otimes id)\Delta = \\ &dE^{1,2} + E^{1,2}(d \otimes id \otimes id + id \otimes d \otimes id + id \otimes id \otimes d). \end{aligned}$$

Now let us interpret this formula in $\mathcal{X}$: it means that the combination

$$W = ((1,2,1) \circ_2 (1,2) + (T \times id)((1,2) \circ_2 (1,2,1) + (1,2) \circ_1 (1,2,1) \in \mathcal{X}(3)_1$$

must be the boundary of some element $U \in E_{1,2} \in \mathcal{X}(3)_2$: $W = dU$. Again, it is easy to discover this element in $\mathcal{X}$.

Let us first calculate this combination:

$$W = ((1,2,1) \circ_2 (1,2) + (T \times id)((1,2) \circ_2 (1,2,1) + (1,2) \circ_1 (1,2,1) =$$
$$(1,2,3,1) + (T \times 1)(1,2,3,2) + (1,2,1,3) =$$
$$(1,2,3,1) + (2,1,3,1) + (1,2,1,3).$$

This is a cycle:

$$dW = (2,3,1) + (1,2,3) + (2,3,1) + (2,1,3) + (2,1,3) + (1,2,3) = 0.$$

Then, again, to find $U_{1,2} \in \mathcal{X}(3)_2$ corresponding to $E_{1,2}$ let us act on the cycle W by the contraction homotopy S and take $U = SW$. Then the condition $dU = W$ will be automatically satisfied.

Let us calculate this element $U = SW$:

$$U = SW = ((1,2,3,1,\mathbf{1}) + (2,1,3,1,\mathbf{1}) + (1,2,1,3,\mathbf{1}) = 0 + 0 + (1,2,1,3,1),$$

thus the operation $E_{1,2}$ is represented by the element $U_{1,2} = (1,2,1,3,1) \in \mathcal{X}(3)_2$.

Now let us check which element of the endomorphism operad

$$End^*(C^*(V)^{\otimes\ *}, C^*(V))$$

corresponds to U, i.e. how this operation acts on $C^*(V)$, or dually on $C_*(V)$

$$(1,2,1,3,1)(v_{i_0}, ..., v_{k_1}, ..., v_{k_2}, ..., v_{k_3}, ..., ..., v_{k_4}, ..., v_{i_n}) =$$
$$\textstyle\sum_{0 \le k_1 < k_2 < k_3 < k_4 \le n}$$
$$(v_{i_0}, ..., v_{i_{k_1}}, v_{i_{k_2}}, ..., v_{i_{k_3}}, v_{i_{k_4}}, ..., v_{i_n}) \otimes (v_{i_{k_1}}, ..., v_{i_{k_2}}) \otimes (v_{i_{k_3}}, ..., v_{i_{k_4}}),$$

and this is exactly the same formula as in [2].

8.7.3 Operation $E_{1,k}$ in $\mathcal{X}$

Let us recall the definition of homotopy G-algebra.

Definition 21 *A homotopy G algebra is a dga $(A, d, \cdot)$ together with a given sequence of multioperations*

$$E_{1,k} : A \otimes A^{\otimes^k} \to A, \quad k = 1,2,3,... \quad ,$$

subject of the following conditions

$degE_{1,k} = -k,\ E_{1,0} = id;$

$$\begin{array}{l} dE_{1,k}(a;b_1,...,b_k) + E_{1,k}(da;b_1,...,b_k) + \sum_i E_{1k}(a;b_1,...,db_i,...,b_k) = \\ b_1E_{1,k-1}(a;b_2,...,b_k) + \sum_i E_{1,k-1}(a;b_1,...,b_ib_{i+1},...,b_k) + \\ E_{1,k-1}(a;b_1,...,b_{k-1})b_k; \end{array} \tag{23}$$

$$\begin{array}{l} a_1E_{1,k}(a_2;b_1,...,b_k) + E_{1,k}(a_1\cdot a_2;b_1,...,b_k) + E_{1,k}(a_1;b_1,...,b_k)a_2 = \\ \sum_{p=1,...,k-1} E_{1,p}(a_1;b_1,...,b_p)\cdot E_{1,k-p}(a_2;b_{p+1},...,b_k); \end{array} \tag{24}$$

$$\begin{array}{l} E_{1,n}(E_{1,m}(a;b_1,...,b_m);c_1,...,c_n) = \\ \sum E_{1,n-\sum l_i+m}(a;c_1,...,c_{k_1},E_{1,l_1}(b_1;c_{k_1+1},...,c_{k_1+l_1}),c_{k_1+l_1+1},...,c_{k_m}, \\ E_{1,l_m}(b_m;c_{k_m+1},...,c_{k_m+l_m}),c_{k_m+l_m+1},...,c_n). \end{array} \tag{25}$$

We claim that each operation $E_{1,k}$ is represented by the following element $U_{1,k} \in \mathcal{X}(k+1)_k$

$$U_{1,k} = (1,2,1,3,1,...,1,k,1,k+1,1).$$

How one can discover this element? By induction, suppose

$$U_{1,i} = (1,2,1,...,1,i+1,1), \quad i < k.$$

Let us rewrite the condition (23) but in terms of elements of $\mathcal{X}$

$$\begin{array}{c} dU_{1,k} = (T\times id\times ...\times id)\cdot(1,2)\circ_2(1,2,1,...,1,k,1) + \\ \sum_{i=2}^k (1,2,1,...,1,k,1)\circ_i(1,2) + (1,2)\circ_1(1,2,1,...,1,k,1). \end{array}$$

Denote by $W_{1,k}$ the right hand side of this equation. So our aim is to find an element $U_{1,k} \in \mathcal{X}(k+1)_k$ satisfying $dU_{1,k} = W_{1,k}$.

Firstly let us calculate the element $W_{1,k}$. Start with the first term

$$\begin{array}{c} (T\times id\times ...\times id)\cdot(1,2)\circ_2(1,2,1,...,1,k,1) = \\ (T\times id\times ...\times id)\cdot(1,2,3,2,...,2,k+1,2) = (2,1,3,1,...,1,k+1,1). \end{array}$$

Now we calculate the second sum

$$\begin{array}{c} \sum_{i=2}^k (1,2,1,...,1,i,1,i+1,1,...,1,k,1)\circ_i(1,2) = \\ \sum_{i=2}^k (1,2,1,...,1,\mathbf{2},\mathbf{3},1,i+2,1,...,1,k+1,1), \end{array}$$

note that all the summands begin with 1.

Finally let us calculate the last term

$$(1,2)\circ_1(1,2,1,...,1,k,1)=(1,2,1,3,1,...,1,k,1,k+1),$$

again note that this term begins with 1.

Finally

$$W_{1,k}=(2,1,3,1,...,1,k+1,1)+(1,2,1,3,1,...,1,k,1,k+1)+\\ \sum_{i=2}^{k}(1,2,1,...,1,\mathbf{2},\mathbf{3},1,i+2,1,...,1,k+1,1).$$

The direct checking (which we omit) shows that $W_{1,k}$ is a cycle, and we can obtain $U_{1,k}$ from $W_{1,k}$ acting with the contraction homotopy s, i.e. adding 1 infant of it, so

$$U_{1,k}=(\mathbf{1},2,1,3,1,...,1,k+1,1)+(\mathbf{1},1,2,1,3,1,...,1,k,1,k+1)+\\ \sum_{i=2}^{k}(\mathbf{1},1,2,1,...,1,\mathbf{2},\mathbf{3},1,i+2,1,...,1,k+1,1)\\ =(\mathbf{1},2,1,3,1,...,1,k+1,1)+0+0=\\ (\mathbf{1},2,1,3,1,...,1,k+1,1).$$

It is clear that so defined operations $E_{1,k}=(1,2,1,3,1,...,1,k+1,1)$ satisfy the condition (23). It is a miracle that they automatically satisfy the conditions (24) and (25) too.

8.8 Deligne's Conjecture

The *Delgne's conjecture* states the following:

The Hochschild cochain complex $C^*(A,A)$ of an associative algebra A is equipped with the structure of an algebra over little disk operad.

Here we do not define the little disc operad, we just mention the result of McClure an Smith [26], [25], that this operad is equivalent to the suboperad of the surjection operad $F_2\mathcal{X}\in\mathcal{X}$ generated by the elements

$$(1,2),\ (1,2,1),\ (1,2,1,3,1),\ ...\ ,(1,2,1,3,1,...,1,k,1),\ ...\ .$$

Combining this fact with the above mentioned fact (7) that $C^*(A,A)$ is a homotopy G-algebra [15], [10], that is elements $(1,2)$, which defines the $\smile$ product, and the elements $(1,2,1,3,1,...,1,k,1)$, $k=1,2,3,...$, which define the brace operations $E_{1,k}$, act on $C^*(A,A)$, McClure and Smith deduce the solution of Deligne's conjecture.

8.9 Extended Homotopy G-algebra operations in the surjection operad

Here we just indicate some operations $E^k_{p,q}$ which form extended homotopy G-algebra.

$E^0_{1,q} = (1;2,1,3,...1,q+1;1)$. These elements generate $F_2\chi$ [26] and they determine on $C^*(X)$ a structure of homotopy G-algebra, that is determine the $\smile$ product on the bar construction.

$$\begin{aligned} E^1_{p,q} = \ & (1;p+1,1,p+2,1,...,p+q-1,1,p+q; \\ & 1,p+q,2,p+q,3,...,p;p+q). \end{aligned}$$

These elements determine the $\smile_1$ product on the bar construction.

$$\begin{aligned} E^2_{p,q} = \textstyle\sum_{k=0}^{q-1} \ & (1;p+1,1,p+2,1,...,1,p+k+1; \\ & 1,p+k+1,2,p+k+1,3,...,p+k+1,p; \\ & p+k+1,p,p+k+2,p,...,p+q;p). \end{aligned}$$

These elements determine the $\smile_2$ product on the bar construction.

The further formulas for operations $E^k_{p,q},\ k=0,1,2,3,...$, defining $\smile_i$-s on the bar construction are given in [20].

Latter on this result was generalized by B.Fresse in [6], who extended this result for all elements of the Barratt-Eccles operad.

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Supported by ICTP and by Award No. GNSF/ST06/3-007
of the Georgian National Science Foundation