Walking in the Quarter Plane

Manuel Kauers (RISC)

Walking in the Quarter Plane

Manuel Kauers (RISC)
Doron Zeilberger (Rutgers)

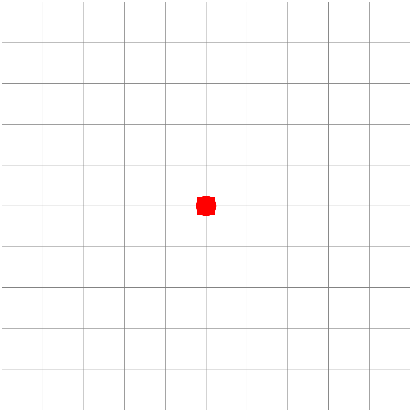
Walking in the Quarter Plane

Manuel Kauers (RISC)
Doron Zeilberger (Rutgers)
Alin Bostan (INRIA)

Walking in the Quarter Plane

Manuel Kauers (RISC)
Doron Zeilberger (Rutgers)
Alin Bostan (INRIA)
Christoph Koutschan (RISC)

Example



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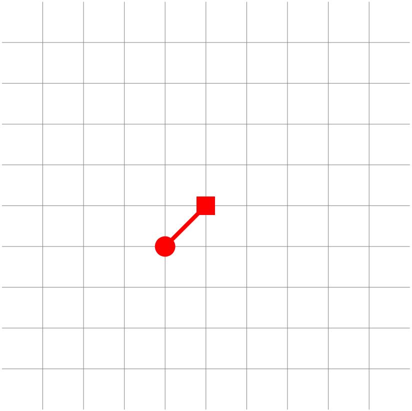

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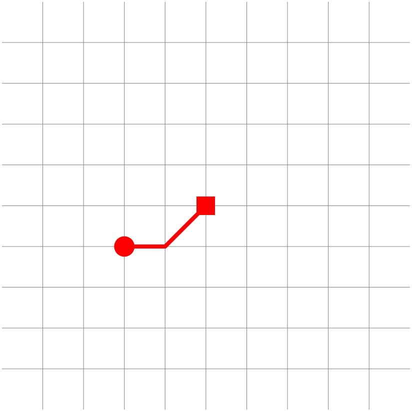

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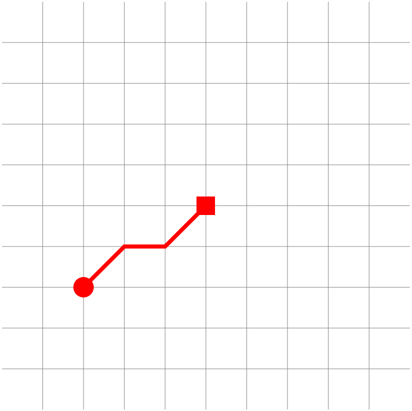

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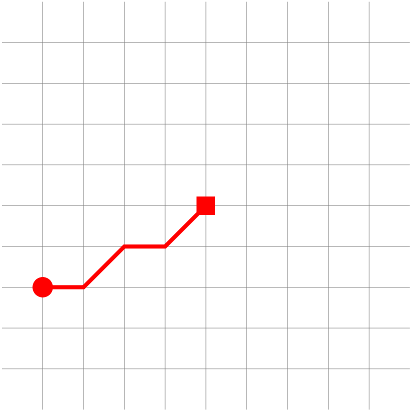

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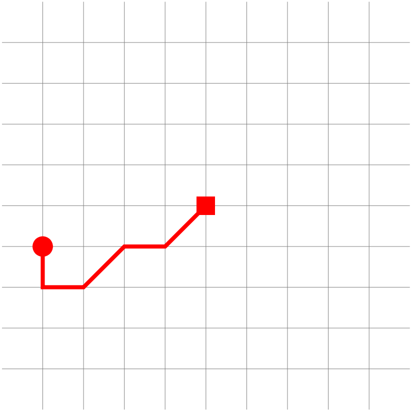

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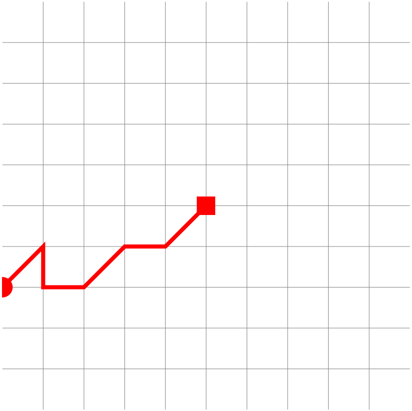

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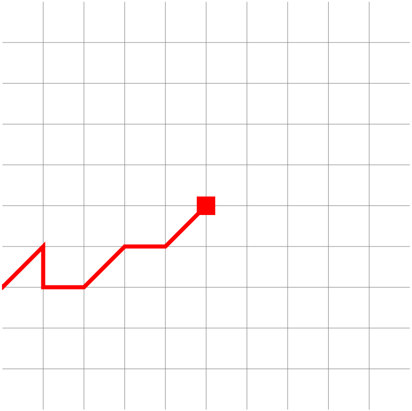

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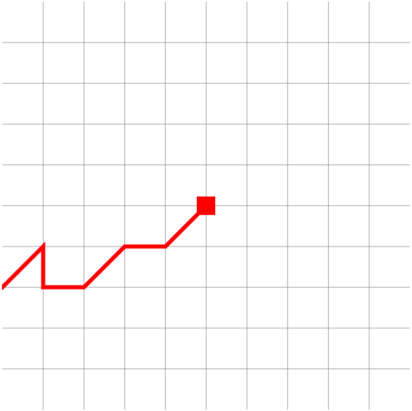

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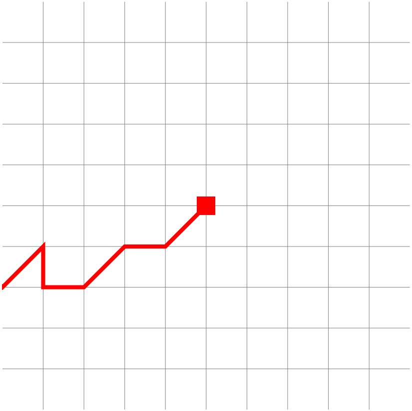

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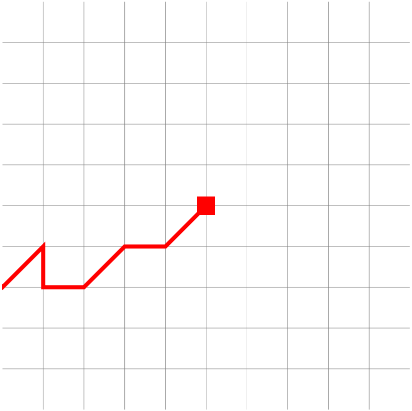

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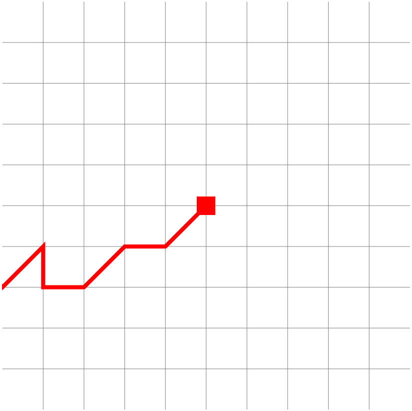

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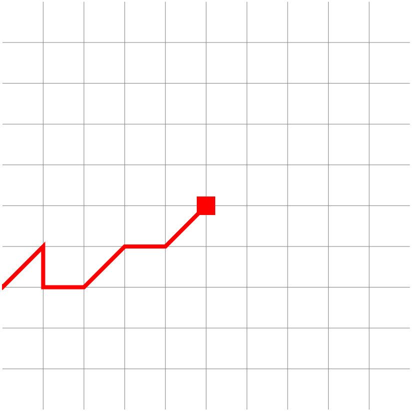

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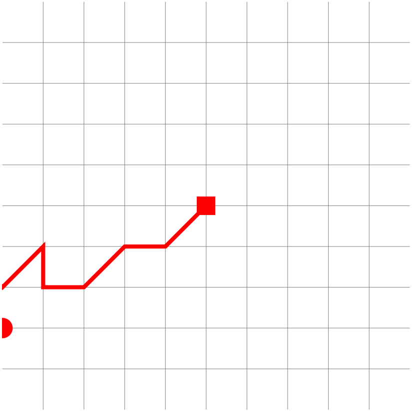

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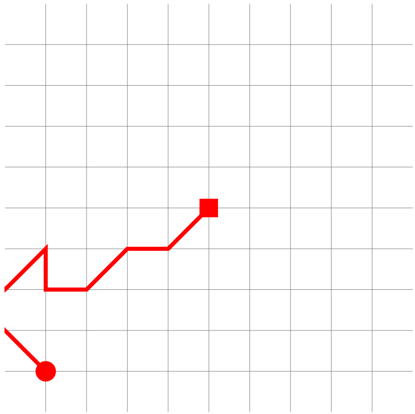

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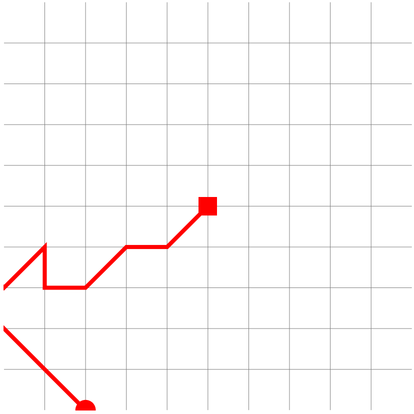

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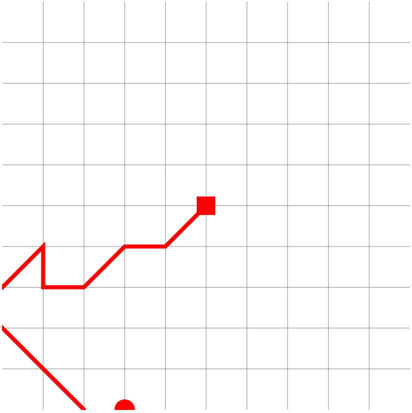

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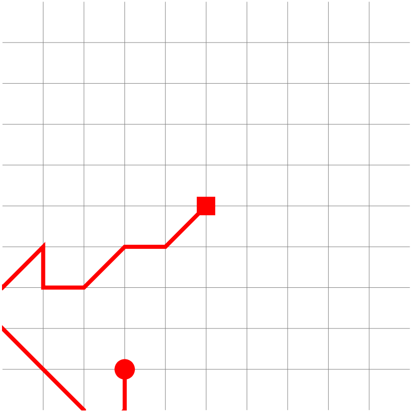

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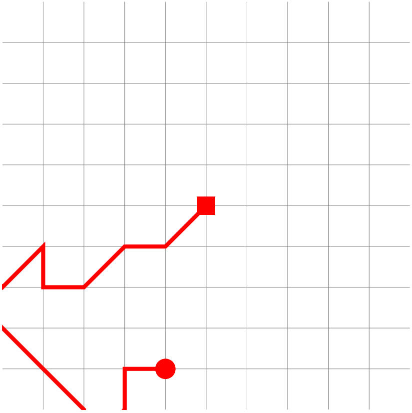

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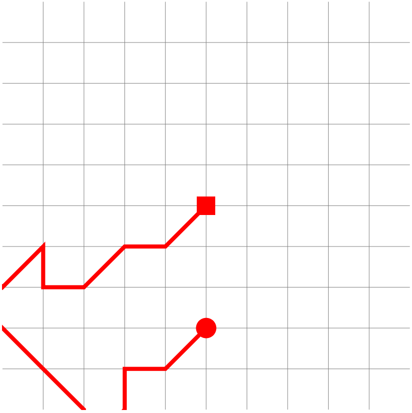

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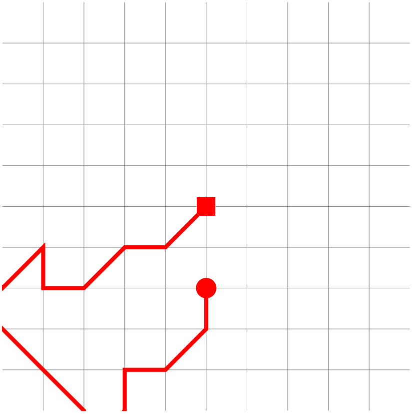

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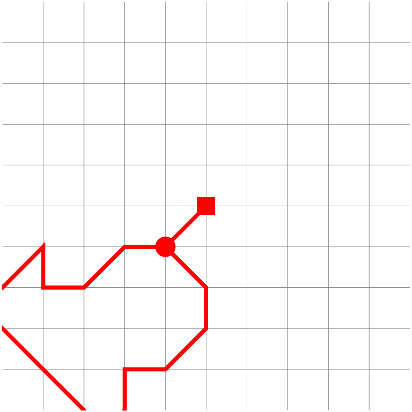

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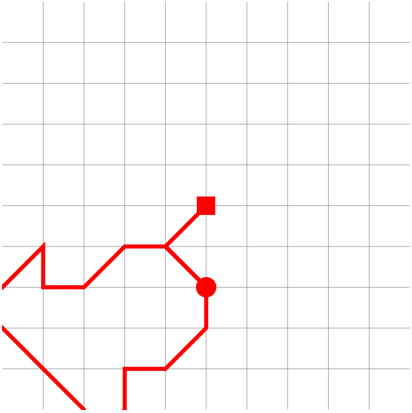

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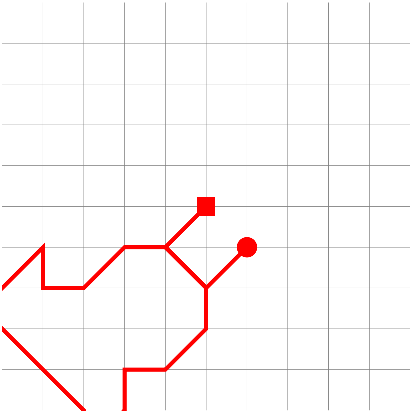

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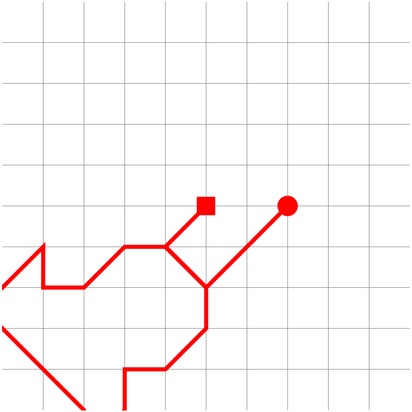

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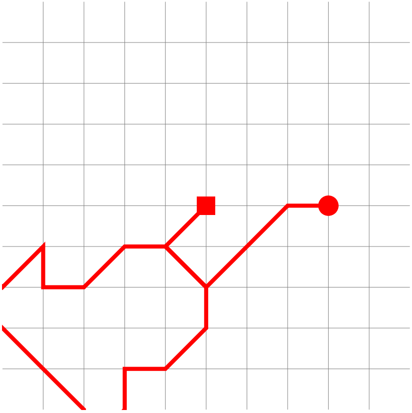

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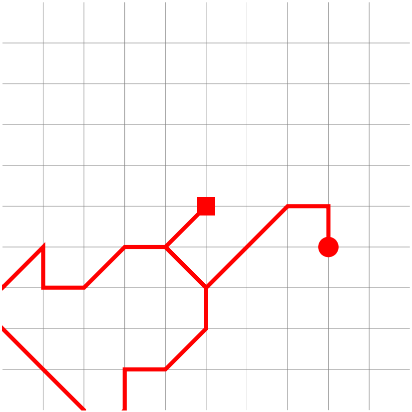

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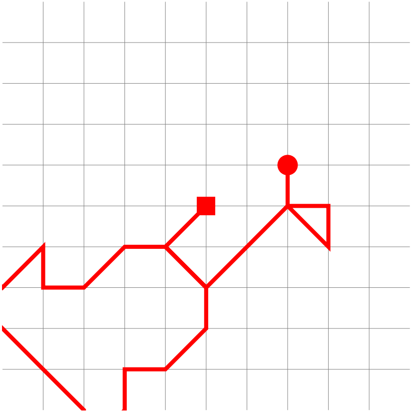

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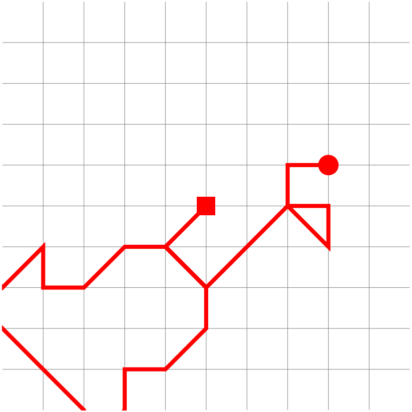

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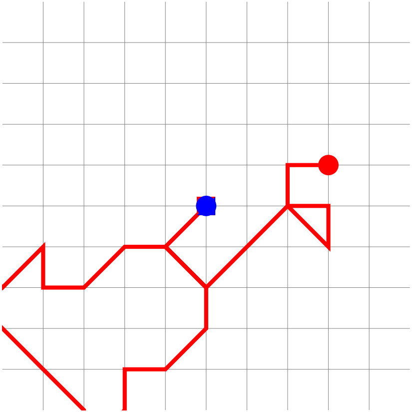

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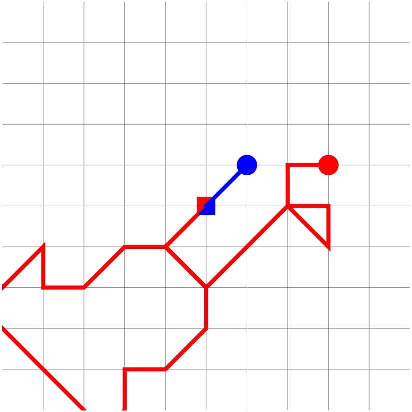

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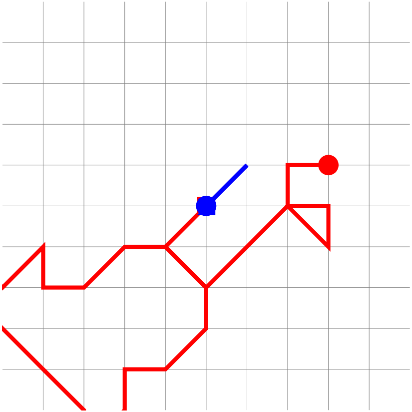

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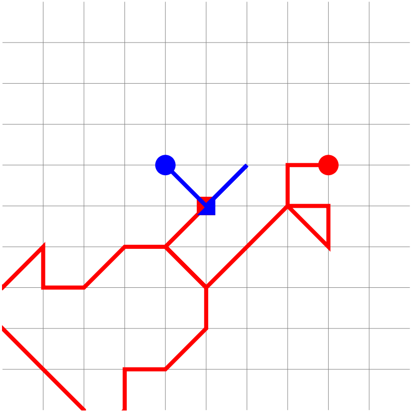

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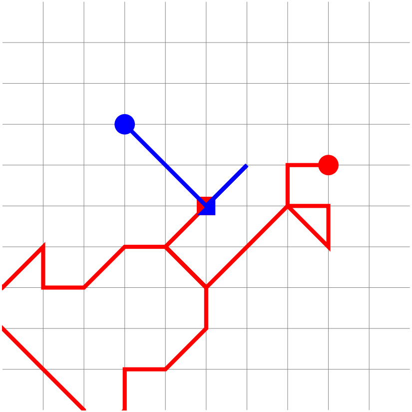

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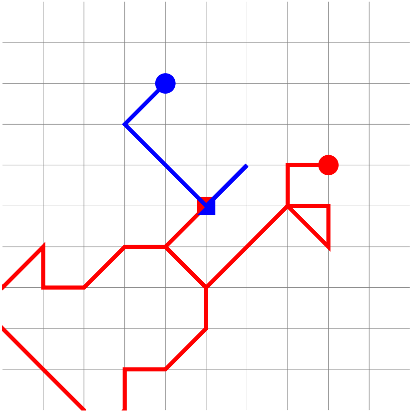

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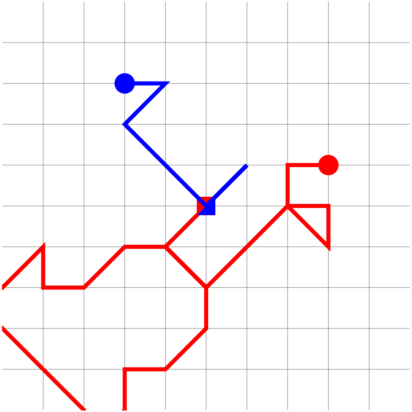

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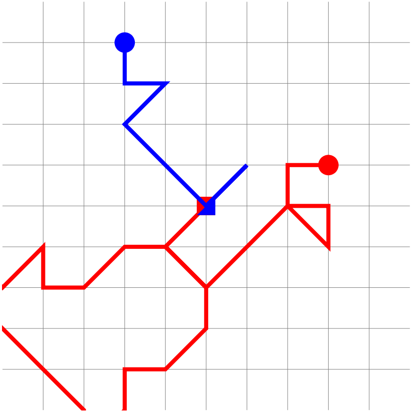

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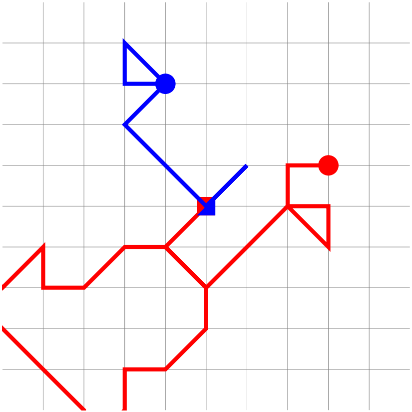

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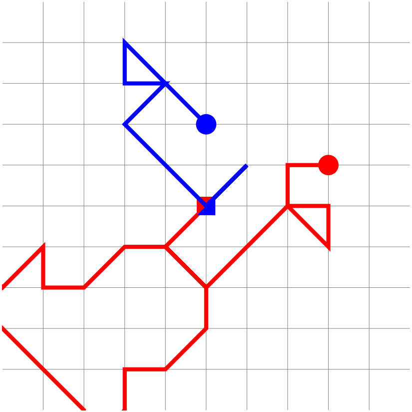

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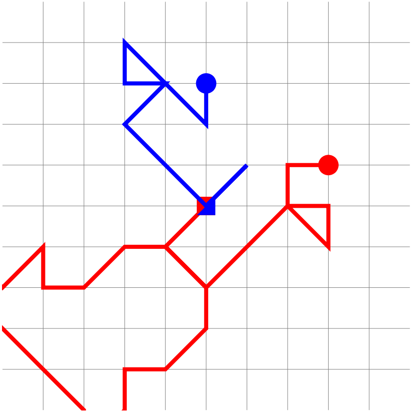

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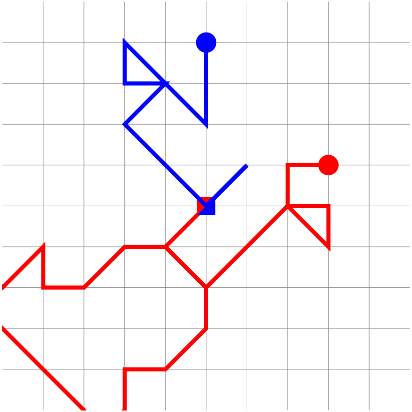

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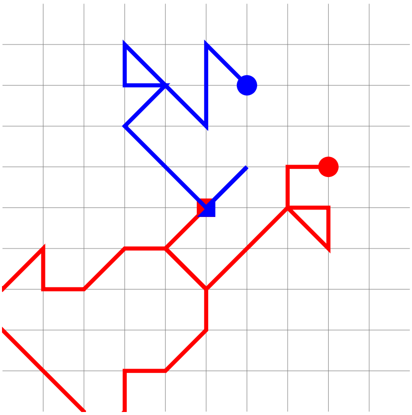

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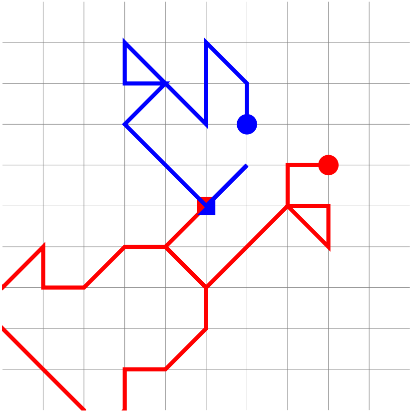

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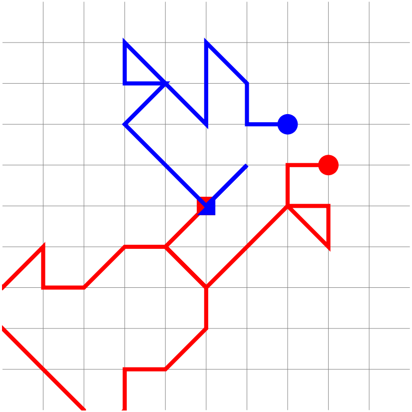

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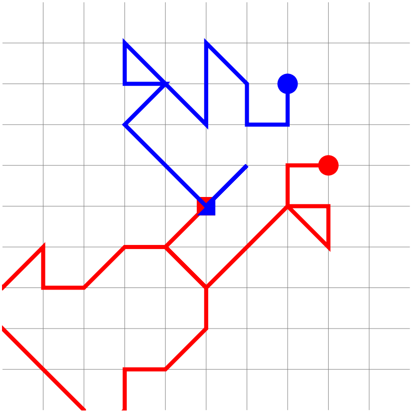

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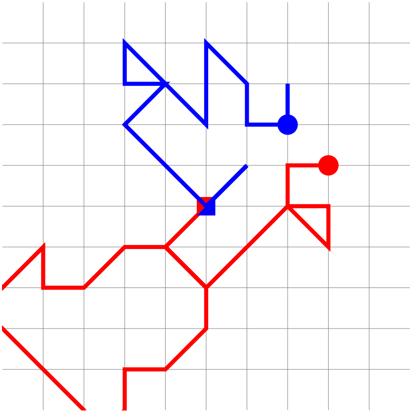

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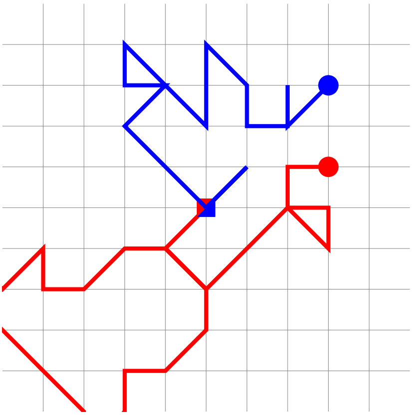

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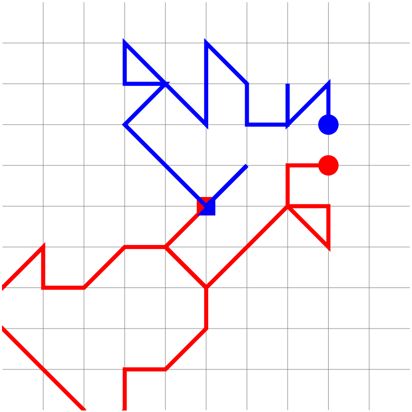

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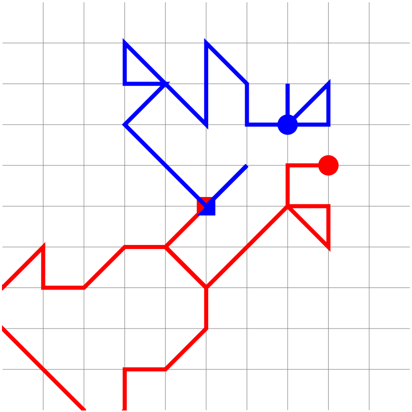

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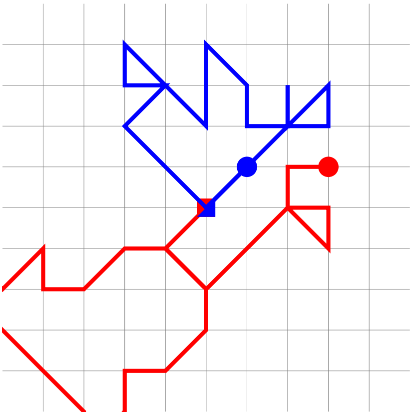

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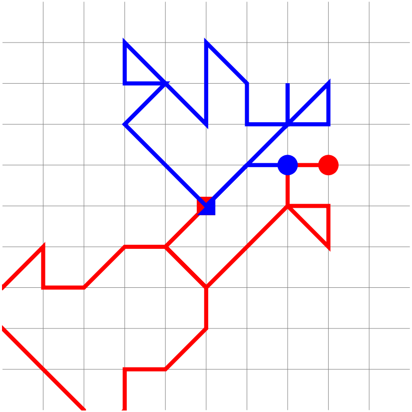

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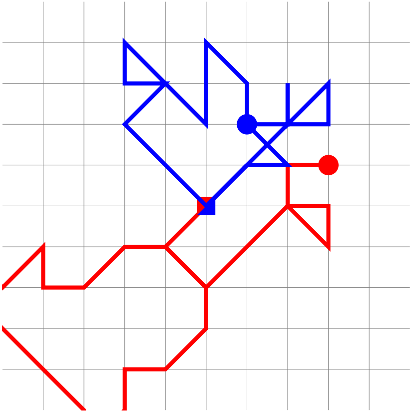

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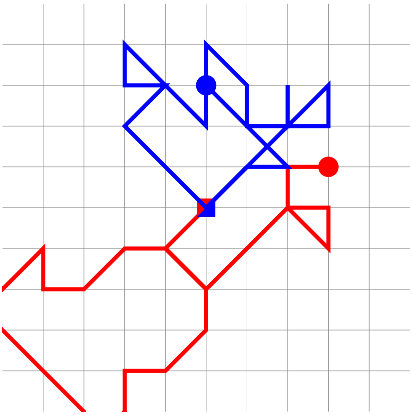

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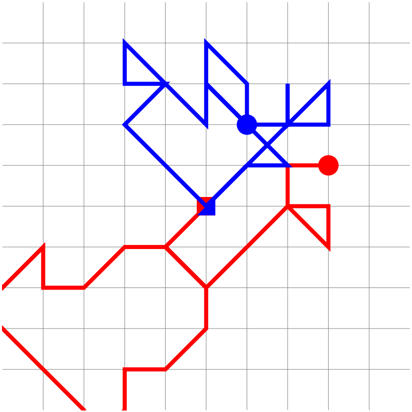

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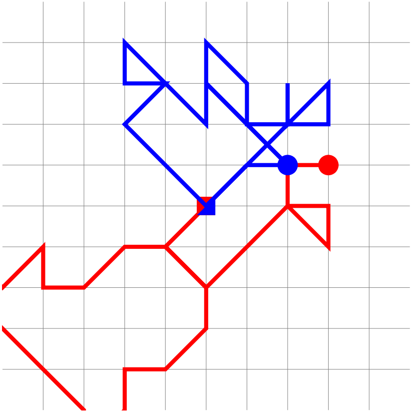

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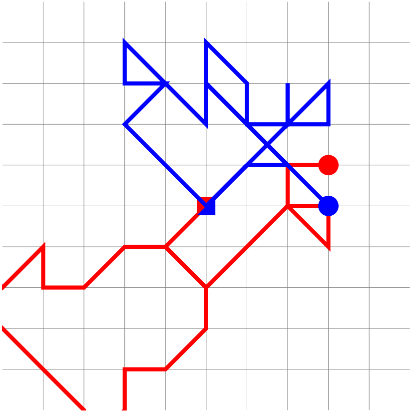

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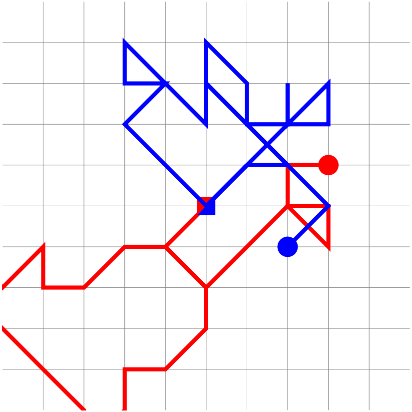

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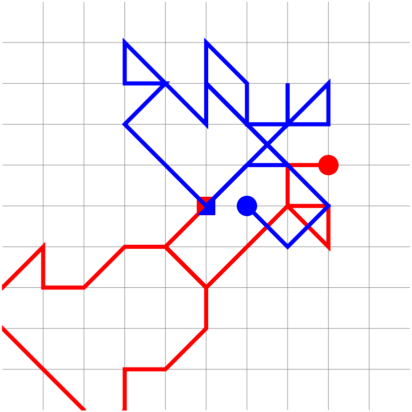

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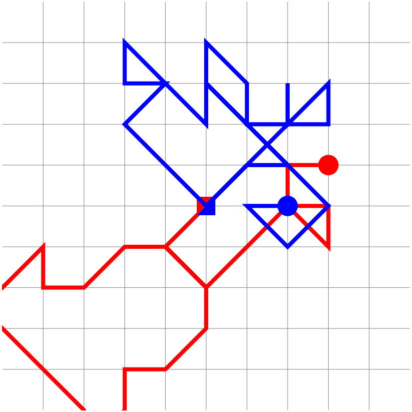

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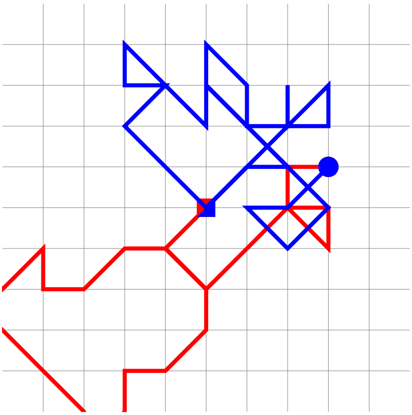

Typical Questions

How many ways are there to walk from $(0,0)$ to $(3,1)$?

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Answer: $17\,604\,317\,873\,070\,171\,384\,276\,000$

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Answer: $3\,018\,900\,111\,360$

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How did I find these numbers?

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Answer: $3\,018\,900\,111\,360$

How did I find these numbers?

How do they depend on the number n of steps?

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How did I find these numbers?

How do they depend on the number n of steps?

How do they depend on the target point (i,j)?

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Answer: $3\,018\,900\,111\,360$

How did I find these numbers?

How do they depend on the number n of steps?

How do they depend on the target point (i,j)?

How are they influenced by restricting the area or the step set?

The Basic Recurrence

Let $f(n; i, j)$ be the number of ways to walk from $(0, 0)$ to (i, j) in exactly n steps, using only $\nearrow$, $\leftarrow$, $\downarrow$.

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Then

$$\begin{aligned} f(n; i, j) = {} & f(n-1; i-1, j-1) \\ & + f(n-1; i+1, j) \\ & + f(n-1; i, j+1) \quad (i, j \in \quad, n \in \quad) \end{aligned}$$

The Basic Recurrence

Let $f(n; i, j)$ be the number of ways to walk from $(0, 0)$ to (i, j) in exactly n steps, using only $\nearrow$, $\leftarrow$, $\downarrow$.

Then

$$\begin{aligned} f(n; i, j) &= f(n-1; i-1, j-1) \\ &+ f(n-1; i+1, j) \\ &+ f(n-1; i, j+1) \quad (i, j \in \quad , n \in \quad) \end{aligned}$$

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Together with the initial condition $f(0;i,j) = \delta_{i,j,0}$, this can be used to compute $f(n;i,j)$ efficiently for fixed n,i,j.

Restricting the walks to the first quadrant amounts to imposing some additional boundary conditions on $f(n;i,j)$.

The Generating Function

$$f(n; i, j)$$

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$$f(n;i,j)x^iy^j$$

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$$\sum_{i,j=-\infty}^{\infty} f(n;i,j)x^i y^j$$

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For unrestricted walks,

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The generating function will be rational for any choice of allowed unit steps.

The Functional Equation

For walks restricted to the first quadrant, the boundary conditions on $f(n; i, j)$ induce into *correction terms* in the equation for the generating function.

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Kreweras' Theorem

Thm. (Kreweras, 1965)
The generating function $F(t; x, y)$ of walks

- inside the first quadrant
- consisting of unit steps $\nearrow$, $\leftarrow$, $\downarrow$

is an (ugly*) algebraic function.

Moreover, $f(3n; 0, 0) = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}$ $(n \geq 0)$.

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The type of $F(t;x,y)$ depends crucially on the step set.

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Mishna's Classification

The type is known for all step sets of cardinality 3 (Mishna, 2007):

steps	gfun is...
$\uparrow, \nearrow, \rightarrow$	rational
$\uparrow, \nearrow, \swarrow$	algebraic, but not rational
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(Other step sets are equivalent to those.)

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Remember: rational $\Rightarrow$ algebraic $\Rightarrow$ holonomic

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Remember: holonomic $\not\Rightarrow$ algebraic $\not\Rightarrow$ rational

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(Notation: $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$.)

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Steps 2. and 3. are routine.

But how to discover a recurrence for $g(2n;0,0)$?

Make an Ansatz!

Make an ansatz

$$(c_0 + c_1 n + c_2 n^2) g(2n; 0, 0) + (c_3 + c_4 n + c_5 n^2) g(2n + 2; 0, 0) = 0$$

with undetermined coefficients $c_0, c_1, c_2, c_3, c_4, c_5$.

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We can *compute* $g(2n; 0, 0)$ for $n = 0, 1, 2, 3, \dots$ and get

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We can *compute* $g(2n; 0, 0)$ for $n = 0, 1, 2, 3, \ldots$ and get

$$c_0 + 2c_3 = 0$$
$$2c_0 + 2c_1 + 2c_2 + 11c_3 + 11c_4 + 11c_5 = 0$$
$$11c_0 + 22c_1 + 44c_2 + 85c_3 + 170c_4 + 340c_5 = 0$$
$$85c_0 + 255c_1 + 765c_2 + 782c_3 + 2346c_4 + 7038c_5 = 0$$
$$782c_0 + 3128c_1 + 12512c_2 + 8004c_3 + 32016c_4 + 128064c_5 = 0$$
$$8004c_0 + 40020c_1 + 200100c_2 + 88044c_3 + 440220c_4 + 2201100c_5 = 0$$

Make an Ansatz!

The solution to this system corresponds to the recurrence

$$(48n^2+128n+80)g(2n;0,0)-(3n^2+22n+40)g(2n+2;0,0)=0.$$

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It might fail for some $n>5$ (although this is *veeery unlikely*.)

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And: They can be verified by an algorithm.

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Operators form a polynomial ring $\quad[n, i, j, N, I, J]$ which is slightly non-commutative. ($Nn = (n+1)N$, etc.)

A recurrence equation corresponds to an annihilating operator

$$P(n, i, j, N, I, J)g(n; i, j) = 0.$$

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This implies termination.

Gessel's Conjecture – 2nd Attempt

Here is a validated annihilating operator for $g(n; i, j)$:

$$\begin{aligned}
&(i-2j+n+2)I^4J^3 + (i-2j+n+2)I^4J^2 \\
&\quad -(i-2j+n+2)I^3NJ^2 - (3j-n-3)I^2J^2 \\
&\quad -(3j-n-3)I^2J + (i+j-1)IJN \\
&\quad -(i+j-1)J - (i+j-1).
\end{aligned}$$

Gessel's Conjecture – 2nd Attempt

Here is the corresponding recurrence:

$$
\begin{aligned}
&(i-2j+n+2)g(n;i+4,j+3) \\
&\qquad + (i-2j+n+2)g(n;i+4,j+2) \\
&\quad - (i-2j+n+2)g(n+1;i+3,j+2) \\
&\qquad - (3j-n-3)g(n;i+2,j+2) \\
&\quad - (3j-n-3)g(n;i+2,j) + (i+j-1)g(n+1;i+1,j+1) \\
&\quad - (i+j-1)g(n;i,j+1) - (i+j-1)g(n;i,j) = 0.
\end{aligned}
$$

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Setting $i = j = 0$ gives

$$\begin{aligned}&(n+2)g(n;4,3)+(n+2)g(n;4,2)\\&\quad-(n+2)g(n+1;3,2)+(n+3)g(n;2,2)\\&\quad+(n+3)g(n;2,0)-g(n+1;1,0)\\&\quad+g(n;0,1)+g(n;0,0)=0.\end{aligned}$$

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This is not very useful, because of the offsets.

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The Cyzak-Salvy-Takayama algorithm can compute P without also computing Q.

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Idea: Apply the Chyzak-Salvy-Takayama Algorithm with i and j in place of $(I-1)$ and $(J-1)$ to find $P(n,N)$ with

$$\big(P(n,N)+iQ_1(n,i,j,N,I,J)+jQ_2(n,i,j,N,I,J)\big)g(n;i,j)=0$$

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At this point it is routine to completing the proof of

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Gessel's Conjectures

Questions:

- $g(2n;0,0) \stackrel{?}{=} 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n}$ for all n?
- Is the generating function

$$G(t;x,y) = \sum_{n=0}^{\infty} \Big(\sum_{i,j=0}^{\infty} g(n;i,j) x^i y^j \Big) t^n$$

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Again, candidate equations can be searched for by *ansatz.*

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Too large to be found by an ansatz with several thousand coefficients.

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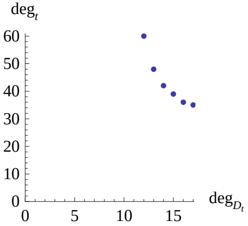
$$P(D_t, t)G(t;\xi,0) = 0$$



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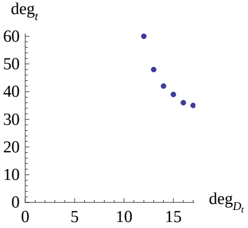

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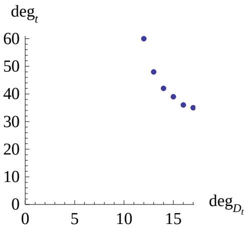

Can an operator $P(D_t, x, t)$ for $G(t; x, 0)$ be interpolated from those?

It seems so, but $\deg_x P$ and the bit size of the integer coefficients will unreasonabley large in the interpolated operator.

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Observe: If P_1, P_2 are annihilating operators of $G(t; \xi, 0)$, then so is $\mathrm{gcrd}(P_1, P_2)$.

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So what...?

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Remember: $G(t;x,y)$ satisfies the functional equation

$$G(t;x,y) = \frac{1}{1 - t(x + \frac{1}{x} + xy + \frac{1}{xy})} \times \left(1 + \frac{1}{xy}\big(G(t;x,0) - G(t;0,0) - (1+y)G(t;0,y)\big)\right)$$

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Therefore: If we believe in the holonomy of $G(t;x,0)$ and $G(t;0,y)$, then we must also believe in the holonomy of $G(t;x,y)$.

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But then there must be also a differential equation for $G(t;x,y)$...

According to estimations, it may have up to $1.5 \cdot 10^9$ terms.

And this is still not all...

Furthermore: It also seems that $G(t; x, 0)$ and $G(t; 0, y)$ are *algebraic.*

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Again, if this is true, then also $G(t; x, y)$ is algebraic, by the functional equation.

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Can we *prove rigorously* that $G(t; x, y)$ is algebraic?

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Can we *prove rigorously* that $G(t; x, y)$ is algebraic?

Yes.

The Kernel Method

Once more:

$$(t + ty - xy + tx^2y + tx^2y^2)G(t;x,y)$$
$$= -xy - tG(t;0,0) + t(1+y)G(t;0,y) + tG(t;x,0)$$

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\end{aligned}
$$

The substitution

$$
\begin{aligned}
y \to y(t,x) &:= \frac{x - t - tx^2 - \sqrt{(t - x + tx^2)^2 - 4t^2x^2}}{2tx^2} \\
&= \tfrac{1}{x}t + \tfrac{1+x^2}{x^2}t^2 + \tfrac{x^4+3x^2+1}{x^3}t^3 + \cdots
\end{aligned}
$$

kills the left hand side and leaves us with

$$
\boxed{G(t;x,0) = G(t;0,0) + y(t,x)x/t - (1 + y(t,x))G(t;0,y(t,x))}
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The substitution

$$x \to x(t,y) := -\frac{y - \sqrt{y^2 - 4yt^2(y+1)^2}}{2ty(y+1)}$$
$$= -\tfrac{y+1}{y}t - \tfrac{(1+y)^3}{y^3}t^3 - \tfrac{2(1+y)^5}{y^5}t^5 + \cdots$$

also kills the left hand side and leaves us with

$$\boxed{(1+y)G(t;0,y) = G(t;0,0) + x(t;y)y/t - F(t;x(t,y),0)}$$

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define the series $G(t;x,0)$ and $G(t;0,y)$ uniquely.

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It can be checked that the *guessed* series satisfy these equations.

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It follows that the guesses were correct. ■.