Certificates of positivity in the multivariate Bernstein basis

Richard Leroy and Marie-Françoise Roy

IRMAR/Université de Rennes 1

d degree, k number of variables, τ bitsize

Certificate of posivity of a polynomial P on a simplex: algebraic identity making it visible that P is indeed positive.

1 Multivariate Bernstein basis

2 How to define the control polytope

3 Certificates of positivity on a simplex

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1 Multivariate Bernstein basis

Simplex V defined by $k+1$ linear inequalities $\ell_j \geqslant 0,\ j=0,\ldots,k$, normalized by

$$1=\ell_0+\ldots+\ell_k$$

Multi-index $i=(i_0,\ldots,i_k)$, of degree d, $|\,i\,|=i_0+\ldots+i_k=d$,

$$\mathrm{Bern}_{d,i}(V)=\frac{d!}{i_0!\cdots i_k!}\prod_{j=0}^{k}\ell_j^{i_j}=\binom{d}{i}\ell^i \tag{1}$$

Think of

$$1=\left(\ell_0+\ldots+\ell_k\cdot\right)^d$$

Properties of the Bersntein basis

- takes positive values on V,
- basis of the vector-space of polynomials of degree $\leqslant d$

If $\deg(P) \leqslant d$ and i a multiindex of degree d, denote by $b(P, d, V)_i$ (or simply b_i) the coefficient of $\mathrm{Bern}_{d,i}(V)$ in P and by $b(P, d, V)$ the vector of Bernstein coefficients of P.

The values of P at the vertices of the simplex are given by $b(P, d, V)_{de_j}$, $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, 1 at place j, $j = 0, \ldots, k$.

Example

$k=1, d=2$

$$X^2, 2\ X\,(1-X), (1-X)^2$$

$k=2, d=2$

$$X^2, 2\,X\,Y, Y^2, 2\,X\,(1-X-Y), 2\,Y(1-X-Y), (1-X-Y)^2$$

$k=3,\ d=2$

develop $(X+Y+Z+(1-X-Y-Z))^2$

2 How to define the control polytope

In the univariate case $k=1$, the multi-index $(d-i, i)$ of degree d is identified with i, $i=0, \ldots, d$, and the points $M_i=(i/d, b_i)$ immediately define the control line above $[0,1]$.

Example

$d=3$, P with coefficients $[4,-6,7,10]$ in the Bernstein basis for $[0,1]$

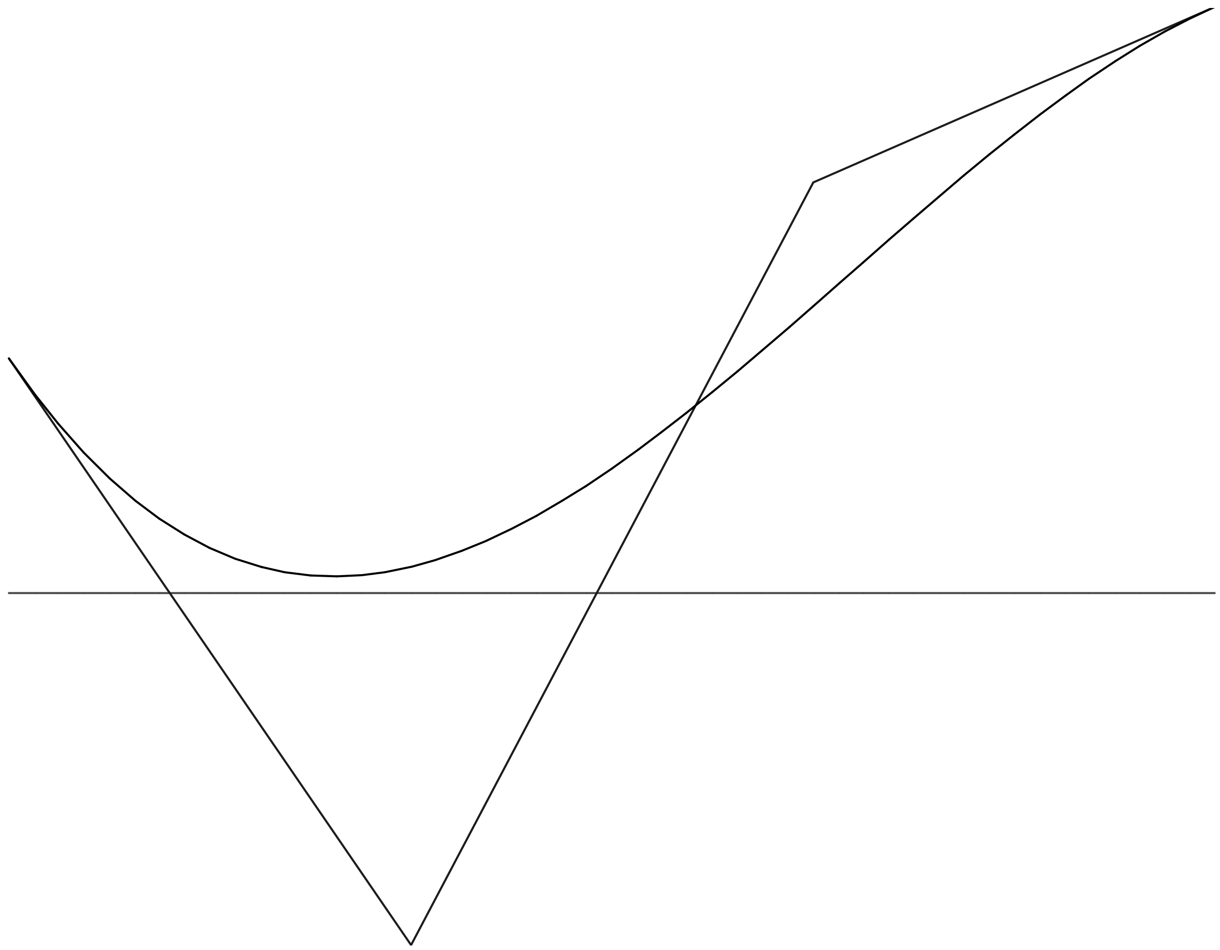

Figure 1. Graph of P and control line of P on $[0,1]$.

In the multivariate case, given a multi-index i of degree d, the points $M_i = (m_i, b_i)$ where m_i is the point of Δ with barycentric coordinates i do not define any more a control polytope above the standard simplex Δ.

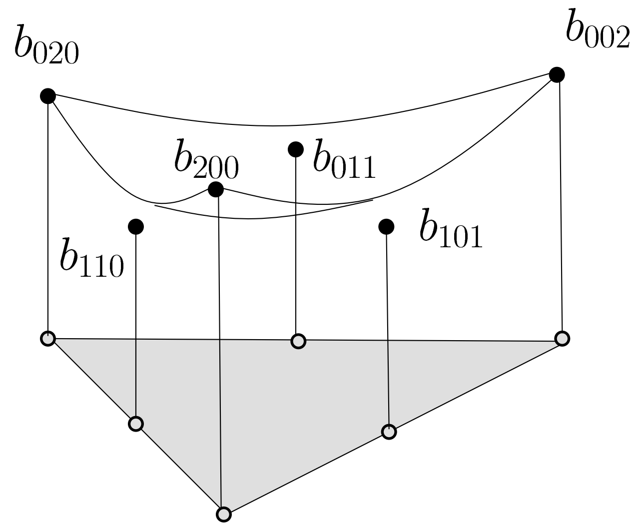

Figure 2. Which control polytope ?

In order to define a control polytope it is needed to define a triangulation of Δ based on the grid points m_i.

1 Standard triangulation [4]

Let V be a simplex with affinely independant vertices v^0, ..., v^k, d the degree. The definition of the standard triangulation $T_{k,d}(V)$ is not intrinsic and depends on the order of the vertices of V.

To every function $F \in \{1, ..., d\}^{\{1,...,k\}}$ is associated a subsimplex V_F of V defined as follows.

Reorder the values of F as

$$f_1 \leq \ldots \leq f_k \,(\text{with } f_0 = 0,\, f_{k+1} = d)$$

Define the multi-index of degree d

$$i_F^0 = (\ldots, f_j - f_{j-1}, \ldots), j = 1, \ldots, k+1$$

an the permutation σ_F of $\{1, \ldots, k\}$

$$\sigma_F(j) = \sharp\{\ell \in \{1, \ldots, k\} \mid F(\ell) < F(j)\} + \ \sharp\{\ell \in \{1, \ldots, j\} \mid F(\ell) = F(j)\}$$

Define, for j from 1 to k multi-indices of degree d

$$i_F^j \ = \ i_F^{j-1} + e_{\sigma_F(j)} - e_{\sigma_F(j)-1}.$$

The simplex $V_F = [v_F^0, \ldots, v_F^k]$ is defined by taking for v_F^j the barycenter of $v_0, \ldots, v_k$ with weights i_F^j.

Example 1 $(k=2, d=2)$

$F(1)$	$F(2)$	$\sigma_F(1)$	$\sigma_F(2)$	V_F
2	2	1	2	[(2,0,0),(1,1,0),(1,0,1)]
2	1	2	1	[(1,1,0),(1,0,1),(0,1,1)]
1	2	1	2	[(1,1,0),(0,2,0),(0,1,1)]
1	1	1	2	[(1,0,1),(0,1,0),(0,0,2)]

$$i_F^0 = (\ldots, f_j - f_{j-1}, \ldots),\ j = 1, \ldots, k+1$$

$$\sigma_F(j) = \sharp\{\ell \in \{1, \ldots, k\} \mid F(\ell) < F(j)\} + \ \sharp\{\ell \in \{1, \ldots, j\} \mid F(\ell) = F(j)\}$$

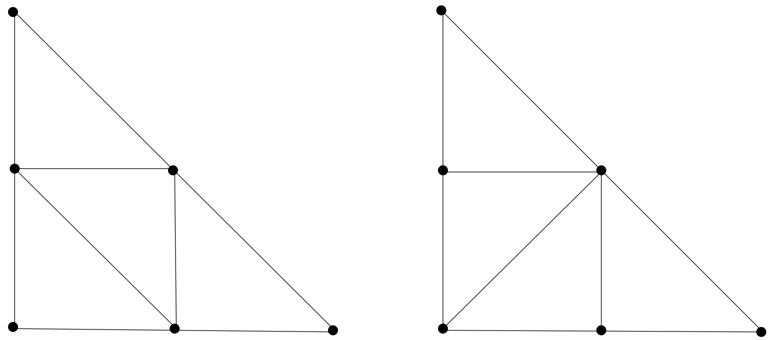

Figure 3. Which is the standard one ?

Example 2: $k=3, d=2$

$F(1)$	$F(2)$	$F(3)$	$\sigma_F(1)$	$\sigma_F(2)$	$\sigma_F(3)$	V_F
2	2	2	1	2	3	[(2,0,0,0),(1,1,0,0),(1,0,1,0),(1,0,0,1)]
2	2	1	2	3	1	[(1,1,0,0),(1,0,1,0),(1,0,0,1),(0,1,0,1)]
2	1	2	2	1	3	[(1,1,0,0),(1,0,1,0),(0,1,1,0),(0,1,0,1)]
1	2	2	1	2	3	[(1,1,0,0),(0,2,0,0),(0,1,1,0),(0,1,0,1)]
2	1	1	3	1	2	[(1,0,1,0),(1,0,0,1),(0,1,0,1),(0,0,1,1)]
1	2	1	1	3	2	[(1,0,1,0),(0,1,1,0),(0,1,0,1),(0,0,1,1)]
1	1	2	1	2	3	[(1,0,1,0),(0,1,1,0),(0,0,2,0),(0,0,1,1)]
1	1	1	1	2	3	[(1,0,0,1),(0,1,0,1),(0,0,1,1),(0,0,0,2)]

Properties of the standard triangulation

- it is a triangulation (simplices intersect along faces)
- it depends on the order of the vertices
- it is invariant under a cyclic permutation of the vertices
- the restriction of $T_{k,d}(V)$ to the simplex V' with vertices v_0, ..., v_r is $T_{r,d}(V')$
- if V_F is a simplex of $T_{k,d}(V)$, $T_{k,\ell}(V_F)$ is the restriction to V_F of $T_{k,d\ell}(V)$.

Control polytope

Once the standard triangulation $T_{k,d}(V)$ of V is defined, it makes sense to define the control polytope of a polynomial P on V: it is the piecewise linear continuous function defined over each $V_F = [v_F^0, \ldots, v_F^k]$ of $T_{k,d}(V)$ by its values at $v_F^j = b(P,d,V)_{i_F^j}$, $j = 0, \ldots, k$.

The control polytope of P is a kind of piecewise linear approximation of the graph of P. The graph of P on V is contained in the convex hull of the control polytope.

Adjacencies in the standard triangulation

Two simplices $V_F = [u^0, ..., u^k]$ and $V_G = [w^0, ..., w^k]$ of the standard triangulation share a common face $[u^0, ..., u^k] \setminus u^j$ in one of the three following cases:

– if $0 < j < k$, then

$$\begin{aligned} F(s) &= G(s), s \neq j, j+1, \\ F(j) &= G(j+1) \\ F(j+1), &= G(j) \end{aligned}$$

– if $j = 0$, then

$$\begin{aligned} F(s) &= G(s+1), s = 1, ..., k-1, \\ F(k) &= G(1) - 1 \end{aligned}$$

– if $j = k$, then

$$\begin{aligned} F(s) &= \mathrm{G}(s-1), s = 2, ..., k, \\ F(1) &= G(k) + 1 \end{aligned}$$

Convexity

Given the adjacency relations between sub-simplexes of $T_{k,d}(V)$, the control polytope of P on V is convex if and only if, with $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, and $e_{-1} = e_k$

$$b_{i+e_j+e_{\ell-1}} + b_{i+e_{j-1}+e_\ell} \geqslant b_{i+e_{j-1}+e_{\ell-1}} + b_{i+e_j+e_\ell}$$

for all $0 \leqslant j < \ell \leqslant k$ and all multi-index i of degree $d-2$.

To P is associated the vector $\delta_2(b)$ whose i, j, ℓ's coordinate is

$$b_{i+e_j+e_{\ell-1}} + b_{i+e_{j-1}+e_\ell} - b_{i+e_{j-1}+e_{\ell-1}} - b_{i+e_j+e_\ell}$$

Example $k=2, d=2$ the vector $\delta_2(b)$ has three components

$$b_{(2,0,0)} + b_{(0,1,1)} - b_{(1,1,0)} - b_{(1,0,1)}$$

$$b_{(0,2,0)} + b_{(1,0,1)} - b_{(1,1,0)} - b_{(0,1,1)}$$

$$b_{(0,0,2)} + b_{(1,1,0)} - b_{(1,0,1)} - b_{(0,1,1)}$$

2.1 Worse possible distance between the graph and the control polytope for the standard simplex

Theorem 1. *The maximum distance between the graph of P and the control polytope of P on the standard simplex Δ is estimated by*

$$\frac{d\,k\,(k+2)}{24}\|\delta_2(b)\|_\infty$$

When $k=1$, classical bound

$$\frac{d}{8}\|\delta_2(b)\|_\infty$$

When $k=2$, bound from [7]

$$\frac{d}{3}\|\delta_2(b)\|_\infty$$

Idea of the proof:

- use convexity and prove that, supposing without loss of generality that $\|\delta_2(b)\|_\infty = 1$, the maximum distance is obtained for a polynomial P such that all components of $\delta_2(P)$ are 1
- construct explicitely a polynomial $P^\star$ such that all components of $\delta_2(b^\star)$ are 1 and compute the difference between the graph and the control polytope for $P^\star$

It turns out that there is a polynomial $P^\star$ of degree 2 such that $\delta_2(b^\star) = 1$. It is the quadratic form associated to the symmetric matrix

$$m_{i,j} = \frac{d\,(d-1)}{2}\, i\,(k - j + 1), i \leqslant j,\ i \leqslant j$$

If $k = 2$ and $d = 2$, we obtain

$$2\,X^2 + 2\ XY + 2\,Y^2$$

which was known to reach the maximum.

For $k > 2$, the result seems to be new.

2.2 Worse possible distance between the graph and the control polytope for the general simplex

Let U be a subsimplex of the standard triangulation of degree 2^N of the standard simplex Δ, and h the diameter of U. We denote by b_i the Bernstein's coefficients of P on Δ and b_i' the Bernstein coefficients of P on U

Theorem 2.

$$\|\delta_2(b')\|_\infty \leqslant \frac{k\,(k+1)(k+2)(k+3)}{24} \left\|\delta_2(b)\right\|_\infty h^2$$

Since $h \leqslant \frac{\sqrt{k}}{2^N}$, we obtain

Theorem 3. *The maximum distance between the graph of P and the control polytope of P on a subsimplex of the standard triangulation is estimated by*

$$\frac{d\,k^3\,(k+1)\,(k+2)^2\,(k+3)}{24^2\,2^{2N}} \|\delta_2(b)\|_\infty$$

3 Certificates of positivity on a simplex

Suppose that P is positive on V. By a certificate of positivity we mean an algebraic identity proving that P is positive on V. There are two kinds of certificates of positivity in the Bernstein basis:

Global certificates of positivity

Express P in the Bernstein basis for increasing degree D. If D is big enough, all the coefficients are positive.

We denote by m the minimum of P on Δ.

Theorem 4. *If P is positive on Δ*

$$D > \frac{d\ (d-1)k(k+2)}{24\, m}\|\delta_2(b)\|_\infty$$

ensures that all the elements of $b(P, D, \Delta)$ are positive.

Different from the bound by Powers and Reznik, sometimes better sometives worse.

Local certificates of positivity

Keep the degree d and subdivide Δ in subsimplices for which all the coefficients of P are positive.

Theorem 5. *If P is positive on Δ*

$$2^N > \frac{\sqrt{d}k(k+2)\sqrt{k\,(k+1)\,(k+3)}}{24\,\sqrt{m}}\sqrt{\|\delta_2(b)\|_\infty}$$

ensures that all the elements of $b(P, D, V_i)$are positive for V_i a simplex of the standard triangulation $T_{2^N}(\Delta)$.

Local certificates are better for two reasons

– the size of the certificates is smaller,

– the process is adaptative, since some simplices do not need to be subdivided.

Subdividing

Algorithm 1. (multivariate De Casteljau)

Input: $(V,\, b(P,p,V))$ and v a barycenter of the vertices with weight $\beta=(\beta_0,\ldots,\beta_k)$

- **Output:** $V_0,\ldots,V_k$ the $k+1$ simplices after subdivision, $b(P,p,V_j)$, $j=0,\ldots,k$.
- **Procedure:**
  - Initialization: $c_i^{(0)} := b(P,p,V)_i$, for $i=(i_0,\ldots,i_k)$, $i_0+\ldots+i_k=p$.
  - For $\ell=1,\ldots,p$,
    - Let $e_j=(0,\ldots,0,1,0,\ldots,0)$, 1 at place number j among $k+1$ numbers.
    - For $i=(i_0,\ldots,i_k)$, $i_0+\ldots+i_k=p-\ell$, compute $c_i^{(\ell)} := \sum_{s=0}^{k} \beta_j\, c_{i+e_j}^{(\ell-1)}$.
    - Output Bersntein's coefficients on V_j $b(P,p,V_j)_i = c_{i-i_j e_j}^{(i_j)}$.

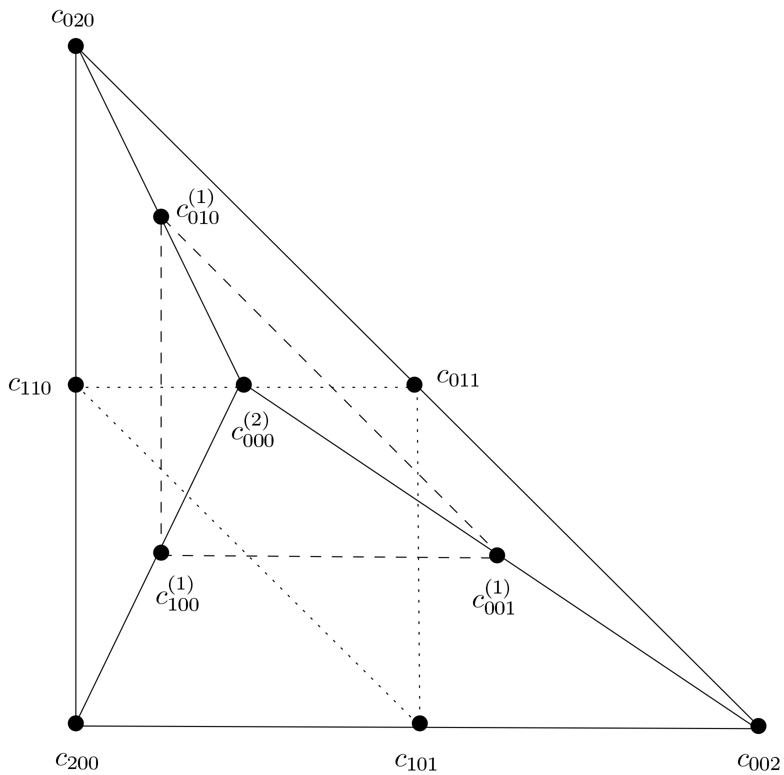

Figure 4.

The only remaining question is: if P is not everywhere positive, how to be sure that the algorithm stops ?

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4 Estimating the smallest possible minimum of a positive polynomial

Suppose P is positive on the unit simplex Δ, is it possible to express as a function $m(d,k,\tau)$ the smallest possible minimum of P on Δ?

This is a natural question in itself.

It also provides a test for ensuring that a polynomial P is not positive on V in the multivariate case:

if the diameter of the subdivision is small enough so that the distance between the graph and the control polytope is at most $m(d,k,\tau)/2$

and

if it is not the case that all the coefficients of P are positive

then P is not everywhere positive.

The principle is simple: suppose that the minimum of P of Δ is obtained in the interior of a face σ of Δ, which is itself a unit simplex of lower dimension. Then the minimum of P on Δ coincides with the minimum of $P_{|\sigma}$ on σ. So it is not a loss of generality to suppose that the minimum of P is obtained in the interior of Δ.

Thus it is sufficient to estimate the value of P on a connected component of the algebraic set Z defined by the zero of $\mathrm{grad}(P)$

$$\frac{\partial P}{\partial X_1} = \ldots = \frac{\partial P}{\partial X_k} = 0$$

intersected with Δ.

In non degenerate situations, the algebraic set Z has a finite number of points. It may however happen that Z has an infinite number of points. So we rely on an algorithm of [1] to compute a point in every connected component of Z intersected with Δ. The minimum of P on Δ is the minimum of the values of P at such a point.

Theorem 6. *The minimum of P on Δ is estimated by the smallest positive root of a polynomial h of degree at most $(2\,d)^k$.*

The bitsize of the coefficients of f can also be estimated (work in progress). Something like

$$k^2\,(2d)^{2k+4}\,(\tau + \text{terms coming from caries})$$

The value of the minimum can be doubly exponentially small.

Rough description of the Certificate of Positivity Algorithm

Initialize the list L of simplices to inspect with Δ

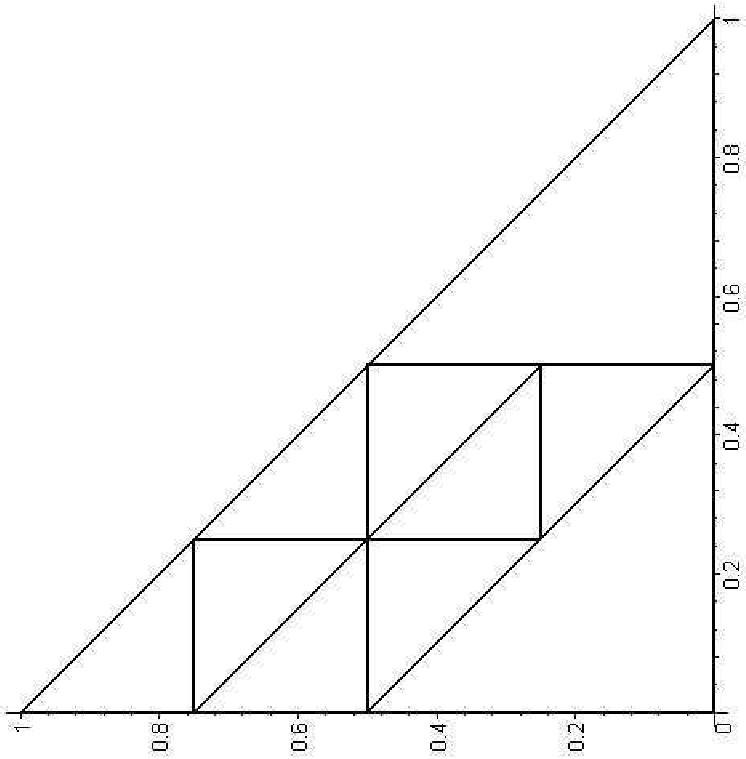
Remove a simplex V from L

If all the elements of $b(P, d, V)$ are positive, store them in a list C

If a value of P at a vertex of V is negative output it

Otherwise subdivide V using the standard triangulation of degree 2, and put all the simplices of $T_2(V)$ in L.

Stop when L is empty OR the diameter of each V on L is small enough to ensure that P is not everywhere positive.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