Spectral Schemes as Ringed Lattices

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joint work, in progress, with

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Partial realisation of Hilbert's programme in algebra:

Work with finite methods and circumvent ideal objects wherever possible.
Prove concrete statements constructively and at their own type level.

Throughout, all rings A will be commutative and all lattices L distributive.

The Zariski Spectrum With Points

The *prime spectrum*

$$\mathfrak{Spec}\,A \equiv \{\mathfrak{p} \subseteq A : \mathfrak{p} \text{ prime ideal of } A\}$$

of A is endowed with the *Zariski topology* whose closed sets are

$$\mathfrak{Z}\,(\mathfrak{a}) \equiv \{\mathfrak{p} \in \mathfrak{Spec}\,A : \mathfrak{a} \subseteq \mathfrak{p}\}$$

where $\mathfrak{a}$ is an ideal of A; the closed points are the maximal ideals of A.

A basis of the Zariski topology is given by the family

$$\mathfrak{D}\,(a) = \mathfrak{Spec}\,A \setminus \mathfrak{Z}\,(a) = \{\mathfrak{p} \in \mathfrak{Spec}\,A : a \notin \mathfrak{p}\} \quad (a \in A)\ .$$

Classically, if $A = k[T_1, \ldots, T_n]$ where k is an algebraically closed field, then the closed points of $\mathfrak{Spec}\, A$ are the elements of k^n, and the traces of the closed subsets $\mathfrak{Z}(\mathfrak{a})$ correspond to the algebraic varieties of k^n.

In general the prime spectra are the local models of the *Grothendieck schemes*, which bridge the gap between algebraic number theory and algebraic geometry. Admitting non-closed points is essential, e.g. for $\mathfrak{Spec}\, \mathbb{Z}$.

Moreover, $\mathfrak{Spec}\, A$ is the prototype of a *spectral space*: a topological space which is sober and whose compact opens form a basis closed by finite meets.

The Zariski Spectrum Without Points

Joyal presented $\mathfrak{Spec}\, A$ as the distributive lattice L_A that is
— generated by the symbolic expressions $D(a)$ with $a \in A$, and
— subject to the relations

$$\begin{aligned} &D(a+b) \leqslant D(a) \vee D(b) \\ &D(a) \wedge D(b) = D(ab) \\ &D(0) = 0\,, \quad D(1) = 1 \end{aligned}$$

This L_A is the distributive lattice of the compact opens of $\mathfrak{Spec}\, A$. The elements of L_A are of the form

$$D(a_1, \ldots, a_n) = D\left(a_1\right) \vee \ldots \vee D\left(a_n\right)\ .$$

The Projective Spectrum With Points

The *projective spectrum* of a graded ring $A = \bigoplus_{d \geqslant 0} A_d$ is

$$\mathfrak{Proj}\, A = \{\mathfrak{p} \subseteq A : \mathfrak{p} \text{ homogeneous prime ideal of } A\}$$

with $A = A_0[x_0, \ldots, x_n]$ for suitable $x_0, \ldots, x_n \in A_1$ with $n \geqslant 1$.

The prime example of a graded ring is $k[x_0, \ldots, x_n]$, graded by degree:

$$\mathbb{P}^n_k = \mathfrak{Proj}\, k[x_0, \ldots, x_n]$$

The Projective Spectrum Without Points

We presented $\mathfrak{Proj}\, A$ as the lattice P_A that is
— generated by the symbols $D(a)$ with $a \in A_d$ for some $d > 0$ and
— subject to the relations

$$\begin{gathered} D(a+b) \leqslant D(a) \vee D(b) \\ D(0) = 0 \\ D(ab) = D(a) \wedge D(b) \\ D(x_0) \vee \ldots \vee D(x_n) = 1 \end{gathered}$$

for all admissible $a, b \in A$. The elements of P_A are of the form

$$D(a_1, \ldots, a_n) = D\left(a_1\right) \vee \ldots \vee D\left(a_n\right) .$$

Grothendieck Schemes With Points

A *locally ringed space* $X = (T, \mathcal{O})$ is a topological space T with a sheaf of local rings $\mathcal{O}$.

A *morphism of locally ringed spaces* $(f, \varphi) : X_1 \to X_2$ is a continuous mapping $f : T_1 \to T_2$ with a homomorphism $\varphi : \mathcal{O}_2 \to \mathcal{O}_1 \circ f^{-1}$ of sheaves of local rings (i.e., induces local homomorphisms on the stalks).

An *affine scheme* is of the form $\mathfrak{Spec}\, A$ with a certain sheaf of local rings.

A *(Grothendieck) scheme* is a locally ringed space that is *locally affine*: that is, has an open cover of affine schemes.

Every $\mathfrak{Spec}\, A$ is a Grothendieck scheme, and so is every $\mathfrak{Proj}\, A$.

The schemes form a full subcategory of the locally ringed spaces.

Sheaves on Lattices

Let L be a poset and C a category.

A *presheaf* on L with values in C is a functor $\mathcal{F} : L^{\mathsf{op}} \to C$.

We now assume that L is a lattice, and that C has finite inverse limits.

A *sheaf* on L with values in C is a presheaf $\mathcal{F}$ such that

$$\mathcal{F}(x_1 \vee \ldots \vee x_n) = \varprojlim \left\{ \mathcal{F}(x_i) \to \mathcal{F}\left(x_i \wedge x_j\right) : i \neq j \right\} . \qquad (*)$$

Lemma Let L' be a basis of a lattice L that is closed by finite meets. If $\mathcal{F}'$ is a presheaf on L' such that (*) holds for all $x_1, \ldots, x_n \in L'$ with $x_1 \vee \ldots \vee x_n \in L'$, then there is a "unique" sheaf $\mathcal{F}$ on L with $\mathcal{F}|_{L'} = \mathcal{F}'$.

Proof Choose $x_1, \ldots, x_n \in L'$ with $x = x_1 \vee \ldots \vee x_n$ in L and set

$$\mathcal{F}(x) = \varprojlim \left\{ \mathcal{F}'(x_i) \to \mathcal{F}'\left(x_i \wedge x_j\right) : \ i \neq j \right\} .$$

The $D(a)$ form a basis of L_A (respectively, of P_A) closed by finite meets.

Ringed Lattices

A *ringed lattice* $X = (L, \mathcal{O})$ is a lattice L with a sheaf of rings $\mathcal{O}$.

A *morphism of ringed lattices* $(f, \varphi) : X_1 \to X_2$ is a lattice homomorphism $f : L_1 \to L_2$ with a natural transformation $\varphi : \mathcal{O}_1 \to \mathcal{O}_2 \circ f$.

Spectral Schemes as Ringed Lattices

An *affine scheme* is a ringed lattice of the form

$$\mathrm{Spec}\ A = (L_A, \mathcal{O}_A)$$

with $\mathcal{O}_A$ uniquely determined by

$$\mathcal{O}_A\left(D\left(a\right)\right) = A\left[\frac{1}{a}\right] \ .$$

If A is an integral domain, then

$$\mathcal{O}_A\left(D(a_1, \ldots, a_n)\right) = A\left[\frac{1}{a_1}\right] \cap \ldots \cap A\left[\frac{1}{a_n}\right] \ .$$

A *spectral scheme* is a ringed lattice $X = (L, \mathcal{O})$ which is *locally affine*: that is, there are $x_1, \ldots, x_n \in L$ with $1 = x_1 \vee \ldots \vee x_n$ such that

$$(\downarrow x_i, \mathcal{O}\,|_{\downarrow x_i}) \cong \mathsf{Spec}\,\mathcal{O}\,(x_i)\ .$$

Any finite sequence $x_1, \ldots, x_n$ of this kind is an *affine cover* of X.

Lemma Every affine scheme is a spectral scheme.

Lemma Let $(L, \mathcal{O})$ be a ringed lattice. If $1 = y_1 \vee \ldots \vee y_m$ in L and each $(\downarrow y_i, \mathcal{O}\,|_{\downarrow y_i})$ is a spectral scheme, then $(L, \mathcal{O})$ is a spectral scheme.

Open Subschemes

If $X = (L, \mathcal{O})$ is a spectral scheme, then

$$X \mid_u = (\downarrow u, \mathcal{O} \mid_{\downarrow u})$$

is the *open subscheme* defined by $u \in L$.

The open subschemes of Spec A are of the form

$$\left(\downarrow D(a_1, \ldots, a_n), \mathcal{O}_A \mid_{\downarrow D(a_1, \ldots, a_n)} \right) \ .$$

If $n = 1$, then this is an affine scheme:

$$\left(\downarrow D(a), \mathcal{O}_A \mid_{\downarrow D(a)} \right) \cong \text{Spec}\, A \left[\frac{1}{a} \right] \ .$$

Lemma Every open subscheme of a spectral scheme is a spectral scheme.

The Projective Scheme as a Spectral Scheme

The *projective scheme* Proj A of a graded ring is $(P_A, \mathcal{O})$ with

$$\mathcal{O}\left(D\left(a\right)\right) = A\left[\frac{1}{a}\right]_0 .$$

This Proj A with $A = A_0[x_0, \ldots, x_n]$ has a canonical affine cover:

$$\left(\downarrow D\left(x_i\right), \mathcal{O}|_{\downarrow D\left(x_i\right)}\right) \cong \text{Spec}\, A\left[\frac{1}{x_i}\right]_0 \qquad (0 \leqslant i \leqslant n) .$$

Lemma Every projective scheme is a spectral scheme.

The Scheme of Valuations as a Spectral Scheme

Let K be a field and R a ring with $R \subseteq K$.

The lattice $\mathsf{Val}_R(K)$ is
— generated by the symbols $V(s)$ with $s \in K$
— subject to the relations

$$r \in R \;\Rightarrow\; V(r) = 1$$
$$V(s) \wedge V(t) \leqslant V(s+t) \wedge V(st)$$
$$s \neq 0 \;\Rightarrow\; 1 = V(s) \vee V(1/s)$$

The elements of $\mathsf{Val}_R(K)$ are the finite joins of the $V(s_1) \wedge \ldots \wedge V(s_n)$.

The points of $\mathsf{Val}_R(K)$ are the *valuation rings* V of K over R: that is, the subrings V of K with $R \subseteq V$ and

$$s \in K \setminus \{0\} \Rightarrow s \in V \vee 1/s \in V\,.$$

If $R = k$ is a field, $s \in K$ transcendental over k, and K a finite algebraic extension of $k(s)$, then the valuation rings of K/k are the points of an algebraic curve over k with function field K.

Define a sheaf of rings $\mathcal{O}$ on $\mathsf{Val}_k(K)$ by

$$\mathcal{O}(x) = \{f \in K : x \leqslant V(f)\} \quad (x \in \mathsf{Val}_k(K)) \ .$$

Lemma The ringed lattice $X = (\mathsf{Val}_k(K), \mathcal{O})$ is a spectral scheme.

Proof There is a two-element affine cover:

$$x_1 = V(s) \ , \ x_{-1} = V\left(s^{-1}\right)$$
$$\left(\downarrow x_i, \mathcal{O}\mid_{\downarrow x_i}\right) \cong \mathsf{Spec}\, E\left(s^i\right)$$

where $E\left(s^i\right)$ is the integral closure of s^i in K.

Sheaves of Modules

Let $X = (L, \mathcal{O})$ be a ringed lattice. A sheaf of abelian groups $\mathcal{M}$ on L is an $\mathcal{O}$*–module* on X if every $\mathcal{M}(x)$ is an $\mathcal{O}(x)$–module such that

$$\begin{array}{ccc} \mathcal{O}(x) \times \mathcal{M}(x) & \longrightarrow & \mathcal{M}(x) \\ \downarrow & \circlearrowleft & \downarrow \\ \mathcal{O}(y) \times \mathcal{M}(y) & \longrightarrow & \mathcal{M}(y) \end{array} \qquad (x \geqslant y) \ .$$

A *sheaf of ideals* on X is an $\mathcal{O}$–submodule $\mathcal{I}$ of $\mathcal{O}$.

For each A–module M there is an $\mathcal{O}_A$–module $\widetilde{M}$ on Spec A with

$$\widetilde{M}\,(D\,(a)) = M\left[\frac{1}{a}\right].$$

Now let $X = (L, \mathcal{O})$ be a spectral scheme. In the following we only consider $\mathcal{O}$–modules $\mathcal{M}$ on X which are *quasicoherent*: that is,

$$\mathcal{M}|_{\downarrow x_i} \cong \widetilde{M_i}$$

for an affine cover $x_1, \ldots, x_n$ of X and $\mathcal{O}\,(x_i)$–modules M_i.
The quasicoherent modules on X form an abelian category.
Every quasicoherent $\mathcal{O}_A$–module on Spec A is isomorphic to some $\widetilde{M}$.

Closed Subschemes

If I is an ideal of the ring A and

$$J = \downarrow \{D(a_1, \ldots, a_n) : a_1, \ldots, a_n \in I\}$$

the corresponding ideal of the lattice L_A, then

$$Z(I) = \left(L_A/J, \mathcal{O}_{A/I}\right) \quad \text{with} \quad \mathcal{O}_{A/I}(D(a)) = (A/I)\left[\frac{1}{a}\right]$$

is a *closed subscheme* of Spec (A) with

$$Z(I) \cong \text{Spec } (A/I)\ .$$

Let $X = (L, \mathcal{O})$ be a spectral scheme and $\mathcal{I}$ a quasicoherent sheaf of ideals on X.
Assume that $x_1, \ldots, x_m$ is an affine cover of X such that $\mathcal{I}|_{\downarrow x_k} \cong \widetilde{I_k}$ where I_k is an ideal of $\mathcal{O}(x_k)$. The *closed subscheme* $Z(\mathcal{I})$ of X defined by $\mathcal{I}$ is obtained by glueing the $Z(I_k)$.

Lemma Every closed subscheme of a spectral scheme is a spectral scheme.

Spectral Morphisms

A *spectral morphism* $(f, \varphi) : X_1 \to X_2$ of spectral schemes is a morphism of ringed lattices which is *locally affine*: that is, there is an affine cover $x_1, \ldots, x_n$ of X_1 such that, with $y_i = f(x_i)$ for every i, for each i there is an affine cover $y_{i1}, \ldots, y_{in_i}$ of $(\downarrow y_i, \mathcal{O}_2 |_{\downarrow y_i})$ for which

$$
\begin{array}{ccccc}
\downarrow x_i & \underset{f}{\longrightarrow} & \downarrow y_i & \underset{\pi}{\longrightarrow} & \downarrow y_{ij} \\
\cong & & & & \cong \\
L\left(\mathcal{O}_1\left(x_i\right)\right) & & \circlearrowleft & & L\left(\mathcal{O}_2\left(y_{ij}\right)\right) \\
D\uparrow & & & & \uparrow D \\
\mathcal{O}_1\left(x_i\right) & \underset{\varphi(x_i)}{\longrightarrow} & \mathcal{O}_2\left(y_i\right) & \underset{r}{\longrightarrow} & \mathcal{O}_2\left(y_{ij}\right)
\end{array}
\qquad (i \leqslant n\,,\ j \leqslant n_i)\ .
$$

For a morphism of ringed lattices (f, φ), to be locally affine roughly means that f locally is determined by φ: that is, with appropriate identifications,

$$f \circ D = D \circ \varphi \,.$$

In particular, (f, φ) satisfies a point-free condition classically equivalent to the one that is to be required from a morphism of *locally* ringed spaces.

Lemma The spectral schemes and spectral morphisms form a category.

Lemma If $X = (L, \mathcal{O})$ is a spectral scheme and $u \in L$, then the inclusion $X \to X\,|_u$ of the open subscheme defined by u is a spectral morphism.

Lemma If $X = (L, \mathcal{O})$ is a spectral scheme and $\mathcal{I}$ a quasicoherent sheaf of ideals on X, then the inclusion $X \to Z\,(\mathcal{I})$ of the closed subscheme defined by $\mathcal{I}$ is a spectral morphism.

Proposition (Universal Property of Spec) For each ring A we have

$$\mathsf{Mor}(\mathsf{Spec}\, A, X) \cong \mathsf{Hom}(A, \mathcal{O}\,(1))$$

natural in spectral schemes $X = (L, \mathcal{O})$.

In the case $X = \mathsf{Spec}\, B$ this reads as

$$\mathsf{Mor}(\mathsf{Spec}\, A, \mathsf{Spec}\, B) \cong \mathsf{Hom}(A, B)\,.$$

Example 1 (Unit Circle) For every ring B there is a bijection

$$\mathsf{Mor}\left(\mathsf{Spec}\,\frac{\mathbb{Z}[X,Y]}{\left(X^2+Y^2-1\right)}, \mathsf{Spec}\,B\right) \cong \left\{(x,y)\in B^2 : x^2+y^2=1\right\} .$$

Example 2 (Projective Space) Let $\mathbb{Z}[X_0,\ldots,X_n]$ be graded by degree. For every ring B there is a bijection between

$$\mathsf{Mor}\left(\mathsf{Proj}\,\mathbb{Z}[X_0,\ldots,X_n], \mathsf{Spec}\,B\right)$$

and the B–modules of rank 1 which are direct summands of B^{n+1}.

Characterisation of Spectral Schemes

We characterise the spectral schemes in classical terms by classical means.

A *spectral space* is a topological space X
— which is sober: that is, every nonempty irreducible closed subspace is the closure of a unique point, its generic point;
— whose compact opens form a basis $\mathsf{K}(X)$ that is closed by finite intersection.

In particular, X is a compact T_0–space, and $\mathsf{K}(X)$ is a distributive lattice.

The topological space of a Noetherian Grothendieck scheme is spectral.

A *spectral mapping* is a continuous mapping $F : X_1 \to X_2$ with

$$V \in \mathsf{K}\,(X_2) \Rightarrow F^{-1}\,(V) \in \mathsf{K}\,(X_1)\,.$$

If a Grothendieck scheme X is Noetherian, then the continuous part of every morphism of Grothendieck schemes $X \to Y$ is a spectral mapping.

The spectral spaces with the spectral mappings form a category. Spectral spaces and distributive lattices are equivalent:

$$\begin{array}{ccc} F : X_1 \to X_2 & \dashrightarrow & F^{-1} : \mathsf{K}\,(X_2) \to \mathsf{K}\,(X_1) \\ f^{-1} : \mathsf{Spec}\, L_2 \to \mathsf{Spec}\, L_1 & \dashleftarrow & f : L_1 \to L_2 \end{array}$$

Proposition The category of spectral schemes is equivalent to the full subcategory of Grothendieck schemes with spectral topological spaces.

It is crucial to see that the continuous part of a morphism of Grothendieck schemes whose topological spaces are spectral is a spectral mapping.

The necessary material has already been present since the early days:

A. Grothendieck, J. A. Dieudonné, *Eléments de Géométrie Algébrique I.* Publ. Math. IHES (1960/61), Springer (1971).

In the 1971 Springer edition of EGA I, three items of one section suffice: "Morphismes quasi-compacts et morphismes quasi-séparés" (**1**, 6.1).

We partially adapt them to the terminology of spectral spaces.

Let $f : X \to Y$ be a morphism of Grothendieck schemes.

— f is quasicompact iff it is a spectral mapping.
— If X is quasicompact and Y quasiseparated, then f is quasicompact.
— Y is quasiseparated iff $\mathsf{K}(Y)$ is closed by binary intersection.

"What would have happened if topologies *without* points had been discovered before topologies *with* points, or if Grothendieck had known the theory of distributive lattices?"

G.-C. Rota, *Indiscrete Thoughts.* Birkhäuser (1997), p. 220

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