Confluence as a cut elimination property

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Independence proofs

Prove that a proposition is not provable in a theory Γ

E.g. atomic proposition $P(t_1, ..., t_n)$ not provable in empty theory

A first method: a counter-model

Interprete all predicates by $\emptyset$

A second method: use cut-elimination

The sequent $\vdash P(t_1, ..., t_n)$ has no cut free proof in sequent calculus: hence no proof at all

A third method

Axioms: $0 \times 0 = 0$, $0 \times 1 = 0$, $1 \times 0 = 0$, $1 \times 1 = 1$

The proposition $0 = 1$ is not provable

The system

$$0 \times 0 \longrightarrow 0$$

$$0 \times 1 \longrightarrow 0$$

$$1 \times 0 \longrightarrow 0$$

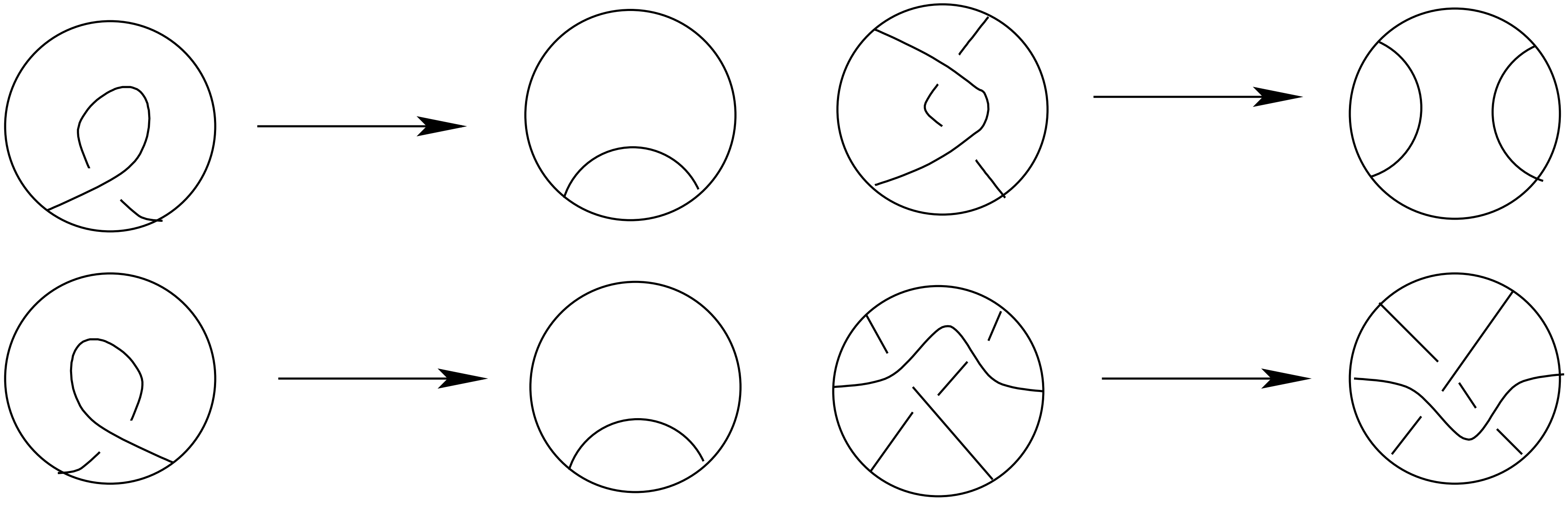
$$1 \times 1 \longrightarrow 1$$

is confluent

Hence $t = u$ provable $\Rightarrow$ t and u have a common reduct.

0 and 1 are normal and distinct: no common reduct

The kind of problem that could be adressed: Reidemeister moves



To prove that two knots and are not equivalent: invariant (model)

An alternative (?): buiding a confluent unknotting rewrite system

What are the relations between these methods ?

Existence of a model and cut elimination

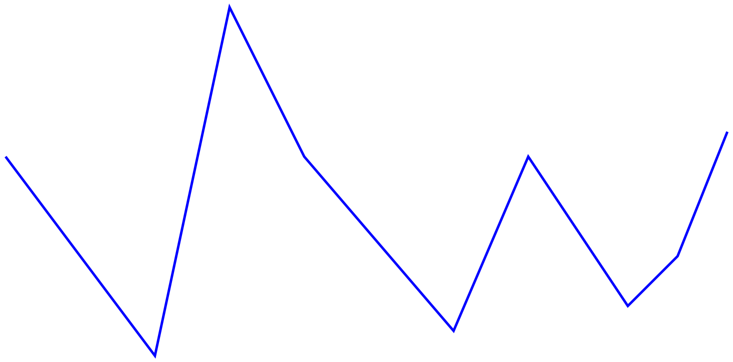
This talk: confluence and cut elimination

Rewriting

A rewrite system on terms

$$l_i \longrightarrow r_i$$

A rewrite sequence $t_1, \dots .t_n$ s.t. $t_i \longrightarrow^1 t_{i+1}$ or $t_i \longleftarrow^1 t_{i+1}$



A peak : $t_{i-1} \longleftarrow^1 t_i \longrightarrow^1 t_{i+1}$

Peaks

(1) A rewrite sequence with no peaks: valley sequence

(2) Contains only reducts of its end-points

(3) Search valley sequence joining t, u: restrict to reducts of t, u

(4) No external information: analytic

(5) If finite number of reducts: decidable

(6) Confluence: all related pairs related by a valley sequence

(7) No valley sequence from x to y: confluence $\Rightarrow$ consistency

Cuts

(1) A proof with no cuts: normal proof

(2) Contains only sub-formulas of its conclusion

(3) Search normal proof of P: restrict to sub-formulas of P

(4) No external information: analytic

(5) If finite number of sub-formulas: decidable

(6) Cut elimination: all provable propositions have a normal proof

(7) No normal proof of $\perp$: cut elimination $\Rightarrow$ consistency

Sequences	Proofs
Peak	Cut
Valley	Normal
Reducts	Sub-formulas
Confluence	Cut elimination

Confluence of a rewrite system $\mathcal{R}$

$=$

Cut elimination of a proof system $\mathcal{R}$'

Deduction modulo

A theory = axioms + rewrite rules

Rewrite rules on terms

$$0 + y \longrightarrow y$$

and on propositions

$$x \times y = 0 \longrightarrow x = 0 \vee y = 0$$

Define a congruence $\equiv$ on propositions

Deduction rules parametrized by $\equiv$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash C, \Delta} \wedge\text{-right} \quad \text{if } C \equiv A \wedge B$$

An example

$$0 + y \longrightarrow y$$

$$S(x) + y \longrightarrow S(x + y)$$

$$0 \times y \longrightarrow 0$$

$$S(x) \times y \longrightarrow x \times y + y$$

$$\cfrac{\cfrac{\overline{4 = 4 \vdash 2 \times 2 = 4}\;\text{Axiom}}{\forall x\ x = x \vdash 2 \times 2 = 4}\;\forall\text{-left}}{\forall x\ x = x \vdash \exists y\ (2 \times y = 4)}\;\exists\text{-right}$$

Let us start slowly

- rules rewriting terms only
- only atomic propositions

Very few rules: Axiom, Cut and structural rules (Weakenning and Contraction)

$$\frac{}{\Gamma, A_1 \vdash A_2, \Delta}\text{Axiom} \quad \text{if } A_1 \equiv A_2$$

$$\frac{\Gamma \vdash C_1, \Delta \quad \Gamma, C_2 \vdash \Delta}{\Gamma \vdash \Delta}\text{Cut} \quad \text{if } C_1 \equiv C_2$$

Cut elimination

Cut elimination: Every provable sequent has a cut free proof

By induction over the structure of derivations: if $\Gamma \vdash \Delta$ provable then $\Gamma \ni A \equiv B \in \Delta$ thus $\Gamma \vdash \Delta$ provable with the axiom rule

Proof normalization: innermost cut

$$\dfrac{\dfrac{}{\Gamma \vdash A, \Delta}\text{Axiom} \qquad \dfrac{}{\Gamma, B \vdash \Delta}\text{Axiom}}{\Gamma \vdash \Delta}\text{Cut}$$

reduces to

$$\dfrac{}{\Gamma \vdash \Delta}\text{Axiom}$$

Cut elimination always holds but...

Are cut free proofs analytic ?

$$\frac{}{\Gamma, A \vdash B, \Delta}\text{Axiom} \quad A \equiv B$$

Yes: no cut no extra information

No: the rewrite sequence from A to B need not be valley

Weaker and weaker: asymmetric deduction modulo

$$\frac{}{\Gamma, A_1 \vdash A_2, \Delta}\text{Axiom} \quad \text{if } A_1 \longrightarrow^* A \longleftarrow^* A_2$$

$$\frac{\Gamma \vdash C_1, \Delta \quad \Gamma, C_2 \vdash \Delta}{\Gamma \vdash \Delta}\text{Cut} \quad \text{if } C_1 \longleftarrow^* C \longrightarrow^* C_2$$

Provability and convertibility

$\Gamma \vdash \Delta$ provable iff $\Gamma \ni A \equiv B \in \Delta$

If $\Gamma \vdash \Delta$ provable then $\Gamma \ni A \equiv B \in \Delta$ induction on the structure of the proof (like in the symmetric case)

If $\Gamma \ni A \equiv B \in \Delta$ then $\Gamma \vdash \Delta$ provable

Induction on the length of the rewrite sequence

If valley: axiom rule, otherwise peak C, induction hypothesis: proofs π and π' of $\Gamma \vdash C, \Delta$ and $\Gamma, C \vdash \Delta$

$$\dfrac{\dfrac{\pi}{\Gamma \vdash C, \Delta} \quad \dfrac{\pi'}{\Gamma, C \vdash \Delta}}{\Gamma \vdash \Delta}\ \text{Cut}$$

Confluence and cut elimination

$\Gamma \vdash \Delta$ has a proof iff $\Gamma \ni A \equiv B \in \Delta$

$\Gamma \vdash \Delta$ has a cut free proof iff $\Gamma \ni A \longrightarrow^* C \longleftarrow^* B \in \Delta$

$$\frac{}{\Gamma \vdash \Delta}\text{Axiom}$$

Main result: Asymmetric deduction modulo $\mathcal{R}$ has the cut elimination property iff $\mathcal{R}$ confluent

Proof normalization: Newman's algorithm

Innermost cut

$$\dfrac{\dfrac{}{\Gamma \vdash C_1, \Delta}\,\text{Axiom} \qquad \dfrac{}{\Gamma, C_2 \vdash \Delta}\,\text{Axiom}}{\Gamma \vdash \Delta}\,\text{Cut}$$

If one of the proofs does not use C_1 or C_2 replace by an axiom rule, otherwise

$$\begin{tikzcd}
A \in \Gamma \arrow[dr, "*"] & & & C \arrow[dl, "*"'] \arrow[dr, "*"] & & & E \in \Delta \arrow[dl, "*"'] \\
& & C_1 \arrow[dl, "*"'] & & C_2 \arrow[dr, "*"] & & \\
& B & & & & D &
\end{tikzcd}$$

If $B = C$ or $C = D$ then replace by an axiom rule. Otherwise

$$\begin{tikzcd}
A \arrow[dr, "*"] & & & C \arrow[dl, "1"'] \arrow[dr, "1"] & & & E \arrow[dl, "*"'] \\
& & C'_1 \arrow[dl, "*"'] & & C'_2 \arrow[dr, "*"] & & \\
& B & & & & D &
\end{tikzcd}$$

If there is a proposition C' such that $C'_1 \rightarrow^* C' \leftarrow^* C'_2$, then

$$\begin{tikzcd}
A \arrow[dr, "*"] & & & C \arrow[dl, "1"'] \arrow[dr, "1"] & & & E \arrow[dl, "*"'] \\
& & C'_1 \arrow[dl, "*"'] \arrow[dr, "*"] & & C'_2 \arrow[dl, "*"'] \arrow[dr, "*"] & & \\
& B & & C' & & D &
\end{tikzcd}$$

reduce to (fail if not)

$$\dfrac{\dfrac{\dfrac{}{\Gamma', A \vdash C'_1, C'_2, E, \Delta'}\,\mathrm{A} \quad \dfrac{}{\Gamma', A, C'_1 \vdash C'_2, E, \Delta'}\,\mathrm{A}}{\Gamma', A \vdash C'_2, E, \Delta'}\,(C'_1)\ \mathrm{C} \quad \dfrac{}{\Gamma', A, C'_2 \vdash E, \Delta'}\,\mathrm{A}}{\Gamma', A \vdash E, \Delta'}\,(C'_2)\ \mathrm{C}$$

A non deterministic algorithm

The algorithm may fail, but not if $\mathcal{R}$ is locally confluent

The algorithm may loop, but not if $\mathcal{R}$ is terminating

If $\mathcal{R}$ is locally confluent and terminating then proofs normalize and hence cut elimination holds and $\mathcal{R}$ is confluent (Newman's theorem)

When Newman meets Gentzen

Extension to the full sequent calculus

Same result: Asymmetric deduction modulo $\mathcal{R}$ has the cut elimination property iff $\mathcal{R}$ confluent

Reduction of "ordinary" cuts work like in Genzen's theorem

$$\dfrac{\dfrac{\dfrac{\rho_1}{\Gamma, A''_1, B''_1 \vdash \Delta}}{\Gamma, A'_1 \wedge B'_1 \vdash \Delta}\ \wedge\text{-l} \qquad \dfrac{\dfrac{\rho_2}{\Gamma \vdash A''_2, \Delta} \quad \dfrac{\rho_3}{\Gamma \vdash B''_2, \Delta}}{\Gamma \vdash A'_2 \wedge B'_2, \Delta}\ \wedge\text{-r}}{\Gamma \vdash \Delta}\ \text{Cut}$$

Reduction of "axiom" cuts work like in Newman's theorem

Two analyticity criteria

Confluence and cut elimination coincide for term rewrite systems

But not for proposition rewrite system

$$A \longrightarrow B \wedge \neg A$$

Confluence is not enough to have analiticity

The "good" analyticity criterion is cut elimination

Symmetric and asymmetric deduction modulo

Symmetric deduction modulo

Termination and confluence: decidability of $\equiv$

Then cut elimination (disjunction property, witness property, completeness of resolution, ...)

In asymmetric deduction modulo

Termination: decidability of $\longrightarrow^*$

Then cut elimination (includes confluence, but more)