The Status of Postnikov “Invariants”

in

Algebraic Topology

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(DVI-file *www-reachable* @ `www-fourier.ujf-grenoble.fr/~sergerar/`)

Restriction: All the topological spaces are assumed to be simply connected and with homotopy groups of finite type.

Classifying the topological spaces:

up to homeomorphism: Impossible.

up to homotopy equivalence: Interesting.

Aim of Postnikov towers:

1. Standardizing the description of homotopy types.
2. Classifying (?) the homotopy types.

General organisation of Postnikov towers.

1. Elementary ingredient: $K(\pi, n)$ where:

   (a) π = Abelian group of finite type;

   (b) n = integer ≥ 2.

2. Infinite product $K(\pi_2, 2) \times K(\pi_3, 3) \times K(\pi_4, 4) \times \cdots$;

3. Postnikov "invariants" k_n twisting the products:

$$K(\pi_2, 2) \times_{k_3} K(\pi_3, 3) \times_{k_4} K(\pi_4, 4) \times_{k_5} \cdots$$

Think of:

$$
\begin{aligned}
S^2 \times_0 S^1 &= S^2 \times S^1 \\
S^2 \times_1 S^1 &= S^3 \\
S^2 \times_2 S^1 &= P^3(\mathbb{R}) \\
\dots &= \dots
\end{aligned}
$$

A $\boxed{\text{fibration}}$ $S^1 \hookrightarrow S^2 \times_n S^1 \to S^2$ is classified by a well-defined Euler class $e \in H^2(S^2, \mathbb{Z})$ because $BS^1 = K(\mathbb{Z}, 2)$.

Warning: But $S^2 \times_1 S^1 = S^3$ and $S^2 \times_{-1} S^1 = S^3$ have the same $\boxed{\text{homotopy type}}$.

Simplest example.

$\pi_2 = \pi_3 = \mathbb{Z}, \qquad \pi_n = 0$ for $n \geq 4$.

$\Rightarrow \quad X = K(\mathbb{Z},2) \times_{k_3 \in ?} K(\mathbb{Z},3)$

Then $k_3 \in H^4(K(\mathbb{Z},2),\mathbb{Z}) = \mathbb{Z}$.

k_3	X_3	H_2	H_3	H_4	H_5	H_6	H_7
0	$K(\mathbb{Z},2) \times_0 K(\mathbb{Z},3)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2 + \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_6$
1	$K(\mathbb{Z},2) \times_1 K(\mathbb{Z},3)$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_6$
2	$K(\mathbb{Z},2) \times_2 K(\mathbb{Z},3)$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 + \mathbb{Z}_2$	0	$\mathbb{Z}_2 + \mathbb{Z}_6$
3	$K(\mathbb{Z},2) \times_3 K(\mathbb{Z},3)$	$\mathbb{Z}$	$\mathbb{Z}_3$	0	$\mathbb{Z}_2 + \mathbb{Z}_3$	0	$\mathbb{Z}_3 + \mathbb{Z}_6$
n	$K(\mathbb{Z},2) \times_n K(\mathbb{Z},3)$	$\mathbb{Z}$	$\mathbb{Z}_n$	0	$\mathbb{Z}_2 + \mathbb{Z}_n$	0	$\mathbb{Z}_n + \mathbb{Z}_6$

Definition of a Postnikov tower: $T = (\pi_n, k_n)_{n\geq 2}$ where:

1. π_n = **Abelian group of finite type**;
2. $k_n \in H^{n+1}(X_{n-1}, \pi_n)$ **where X_n defined below.**

X_n defined by a recursive process:

1. $X_1 = * =$ **one point space;**
2. **If X_{n-1} available, then $k_n \in H^{n+1}(X_{n-1}, \pi_n)$**
   and $X_n = X_{n-1} \times_{k_n} K(\pi_n, n)$.

		$\boldsymbol{X_1 = *}$
$\boldsymbol{\pi_2}$	$\boldsymbol{k_2 = 0}$	$\boldsymbol{X_2 = K(\pi_2, 2)}$
$\boldsymbol{\pi_3}$	$\boldsymbol{k_3 \in H^4(X_2)}$	$\boldsymbol{X_3 = K(\pi_2, 2) \times_{k_3} K(\pi_3, 3)}$
$\boldsymbol{\ldots}$	$\boldsymbol{\ldots}$	$\boldsymbol{\ldots}$
$\boldsymbol{\ldots}$	$\boldsymbol{\ldots}$	$\boldsymbol{X_{n-1} = K(\pi_2, 2) \ldots \times_{k_{n-1}} K(\pi_{n-1}, n-1)}$
$\boldsymbol{\pi_n}$	$\boldsymbol{k_n \in H^{n+1}(X_{n-1}, \pi_n)}$	$\boldsymbol{X_n = \ldots \times_{k_{n-1}} K(\pi_{n-1}, n-1) \times_{k_n} K(\pi_n, n)}$

EDM quotation (subarticle "The Postnikov Complex"):

> **As yet we are $\boxed{\text{ignorant}}$ of an effective method of computing the cohomology of a Postnikov complex from π_n and k_n.**

In fact now solved thanks to the Effective Homology theory.
Also theoretically solved by Rolf Schön,
but without concrete applications.

Concrete implementation of Effective Homology theory
= Kenzo program.

Used in a previous slide to compute $H_*(K(\mathbb{Z},2) \times_{k_3} K(\mathbb{Z},3))$.

Conversely:

Postnikov's theorem: If X = topological space,

then $\exists\, T_X = (\pi_n, k_n)_{n\geq 2}$ such that $X = \lim T_X$.

So that there exists a "correspondance":

$$\begin{array}{ccc} \text{Topological spaces} & \Leftrightarrow & \text{Postnikov Towers} \\ X & \Leftrightarrow & T_X \end{array}$$

which looks like a classification process.

Question: Does T_X "classify" X?

Question: What a classification process exactly is?

A classification process is a map $\chi : E \longrightarrow C$ satisfying:

1. **The classification map χ is surjective.**
2. **The set C is not too large ($\mathrm{id} : E \longrightarrow E$?!?!).**
3. **The set C is not too small ($* : E \longrightarrow \{*\}$?!?!).**
4. **The induced equivalence relation on E is worthwhile.**
5. **The set C is recursive ($\pi : E \longrightarrow E/S$?!?!).**
6. **The map χ is recursive.**
7. **A recursive section of χ can be given.**

Typical example.

Let K be a given ground field and $\mathcal{E}$ be the class of all the endomorphisms of finite dimensional K-vector spaces.

Then a possible classification process is the pair (χ, C):

$$C = \{(p_1, \ldots, p_k) \in K[X]^k \ \underline{\text{st}}\ p_1 \,|\, p_2 \,|\, \ldots \,|\, p_k\}_{k \in \mathbb{N}}$$

and

$$\chi : \mathcal{E} \to C$$

associates to a matrix its invariant factors.

$\Rightarrow$

Successful classification of endomorphisms up to conjugation.

Hans Baues in a paper "Homotopy Types" (1995):

> **Here k_nX is $\boxed{\text{actually}}$ an invariant of the homotopy type of X in the sense that a map $f : X \to Y$ satisfies:**
> $$(P_nf)^*k_nY = k_nX.$$

Clear translation: if you give me some homotopy equivalence between the spaces X and Y, then I'm able to prove the corresponding "invariants" are... "equivalent"!

But the interesting problem is obviously the $\boxed{\text{inverse}}$ one.

Typical example.

A "simple" Postnikov tower with only:

$$\pi_2 = \mathbb{Z}^d, \quad \pi_5 = \mathbb{Z}.$$

Then only one non-trivial k-"invariant":

$$k_5 \in H^6(X_2, \mathbb{Z}) = H^6(K(\mathbb{Z}^d, 2), \mathbb{Z}) = H^6(P^\infty(\mathbb{C})^d, \mathbb{Z})$$

$$= \mathbb{Z}[t_1, \ldots, t_d]_3.$$

$\Rightarrow \forall P \in \mathbb{Z}[t_1, \ldots, t_d]_3$, a space Y_P is well defined by the Postnikov tower:

$$((\pi_2 = \mathbb{Z}^d, 0), (0, 0), (0, 0), (\pi_5 = \mathbb{Z}, k_5 = P), (0, 0), \ldots)$$

$$Y_P = K(\mathbb{Z}^d, 2) \times_{k_5} K(\mathbb{Z}, 5).$$

Fundamental question.

What about the decision problem: Let:

$$P_1, P_2 \in \mathbb{Z}[t_1, \dots, t_d]_3$$

be given. Have:

$$K(\mathbb{Z}^d, 2) \times_{P_1} K(\mathbb{Z}, 5) \text{ and } K(\mathbb{Z}^d, 2) \times_{P_2} K(\mathbb{Z}, 5)$$

the $\boxed{\text{same}}$ homotopy type?

Amounts to linear equivalence:

$$P_1(t_1, \dots, t_d) \quad \underset{\cong}{\xleftrightarrow{GL(d,\mathbb{Z}) \times GL(1,\mathbb{Z})}} \quad P_2(t_1, \dots, t_d)$$

Analogous problems if $\pi_2 = \mathbb{Z}^d$ and $\pi_{2n-1} = \mathbb{Z}$ with the polynomials $\mathbb{Z}[t_1, \ldots, t_d]_n$.

Many arithmeticians were questionned:

> **What about the $GL(d, \mathbb{Z})$-equivalence of homogeneous $\mathbb{Z}$-polynomials in d variables $t_1, \ldots, t_d$?**

No positive answer: the problem seems completely open.

Corollary: so far the so-called k-invariant is a "non-constructive" invariant.

Organizing the exact nature of Postnikov "invariants".

Theorem: A functorial algorithm PR (= Postnikov realization) can be written down:

$$\mathrm{PR} : \mathcal{PT} \longrightarrow \mathcal{SS}_{EH}$$

In fact necessary for the very definition of a Postnikov tower:

$$\mathcal{PT} \ni T = (\pi_2, k_2, \pi_3, k_3, \ldots, \pi_n, \ k_n \in \ ???)$$

By definition $k_n \in H^{n+1}(X_{n-1}, \pi_n)$,

but the knowledge of X_{n-1} $\boxed{\boxed{\text{needs}}}$ PR_{n-1} !!!

(T, PR) = pair of associated $\boxed{\text{coroutines}}$.

"Inverse" algorithm:

Theorem: An algorithm SP (= Simplicial sets $\longrightarrow$ Postnikov towers) can be written down:

$$\mathrm{SP}\ : \mathcal{SS}_{EH} \to \mathcal{PT} \times \mathcal{I}$$

where $\mathrm{SP}(Y) = (\pi_n, k_n, I_n)$ with:

1. $\pi_n =$ isomorphism class of $\pi_n(Y)$;
2. $k_n \in H^{n+1}(X_{n-1}, \pi_n)$;
3. $I_n : \pi_n(Y) \xrightarrow{\cong} \pi_n$.

I_n is arbitrary but $\boxed{\text{must}}$ be "chosen"!!

Corollary:

The terminology "Postnikov invariant" is erroneous.

Essential nature: confusion between functor and invariant.

The measure of the error is the non-symmetry between the algorithms PR and SP.

Getting rid of this error amounts to solving a general Diophantine problem, the $\mathbb{Z}$-linear equivalence between elements of $\mathbb{Z}[\boldsymbol{t}_1, \ldots, \boldsymbol{t}_d]_n$ being the first particular case

Other point of view:

$$X \overset{k_n}{\longmapsto} k_n(X) \in H^{n+1}(X_{n-1}, \pi_n(X))$$

is Baues' $\boxed{\boxed{\text{functor}}}$, correctly defined.

Transforming k_n into an actual invariant amounts to removing both occurences of $\boxed{X}$ in $H^{n+1}(\boxed{X}_{n-1}, \pi_n(\boxed{X}))$.

Replacing $\boxed{X}_{n-1}$ and $\pi_n(\boxed{X})$ by something without any $\boxed{X}$ amounts to solve a general Diophantine problem.

Main question:

Is it possible to avoid the Matiyasevich obstacle?

Open problem.

$\Longleftrightarrow$

Is the "reasonable" simply connected algebraic topology decidable?

Open problem.

Questions:

1) How to formalize the notion of classification?

2) How to formalize the notion of invariant?

The END

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