A Certificate for Budan’s Theorem in Polynomial Time

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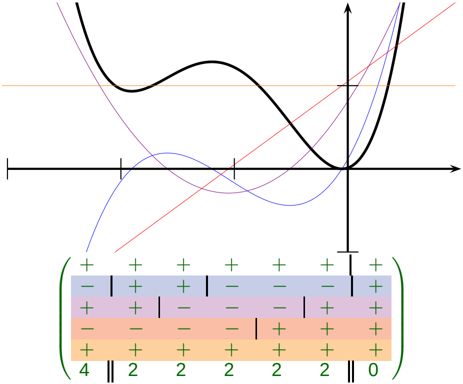
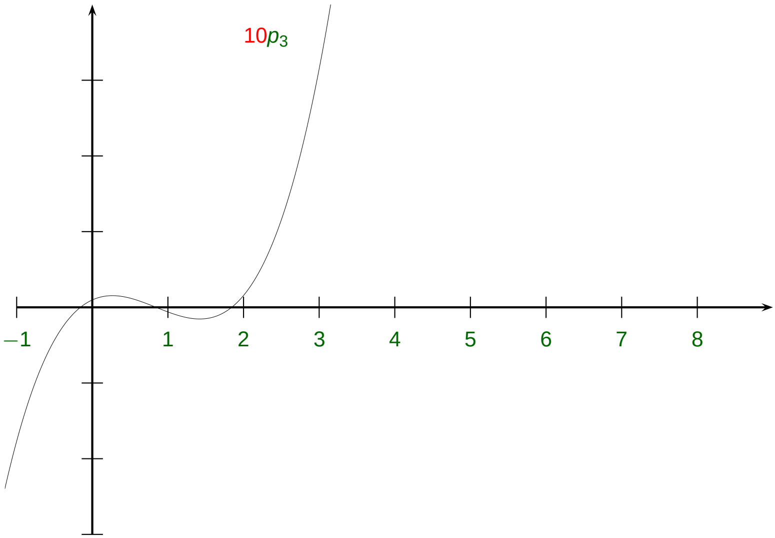
Theorem (Budan 1811)

Let be $f \in \mathbb{R}[X]$ of degree n and $a < b \in \mathbb{R}$. Then

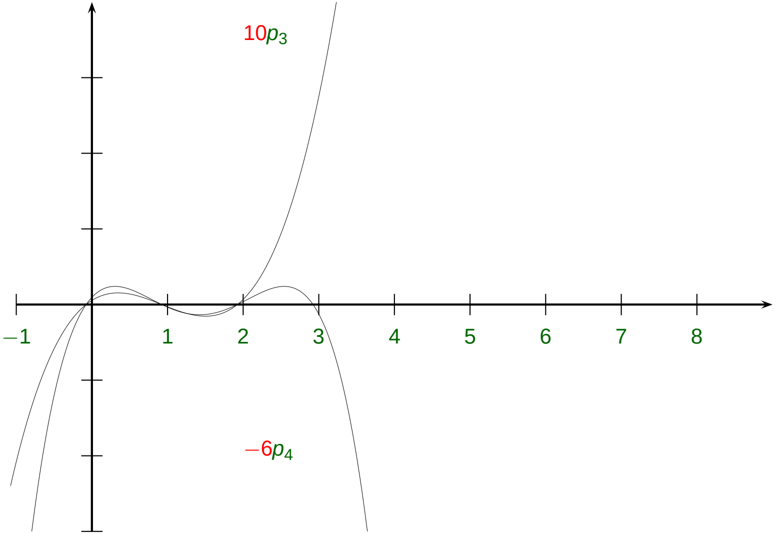
$$Var(f(a), f'(a), \ldots, f^{(n)}(a)) \geq Var(f(b), f'(b), \ldots, f^{(n)}(b))$$



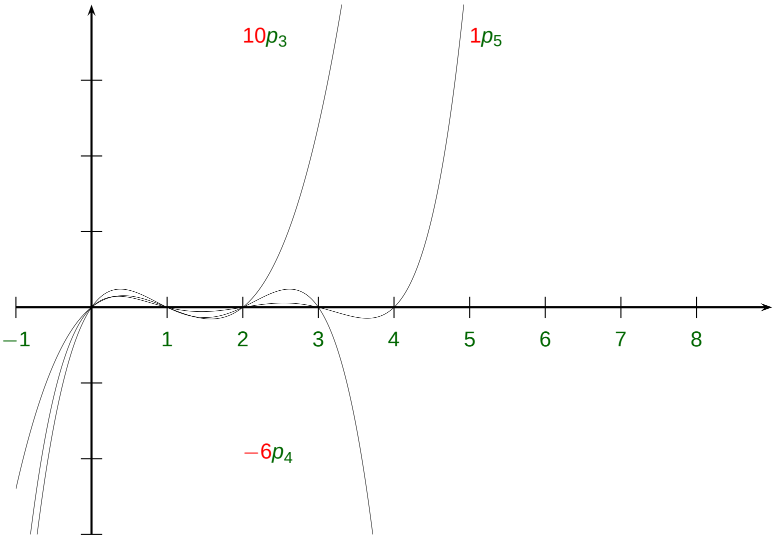
Example (Linear combination $q_{5,8}^{3,4,5}$ of p_3, p_4, p_5 with roots 5 and 8.)



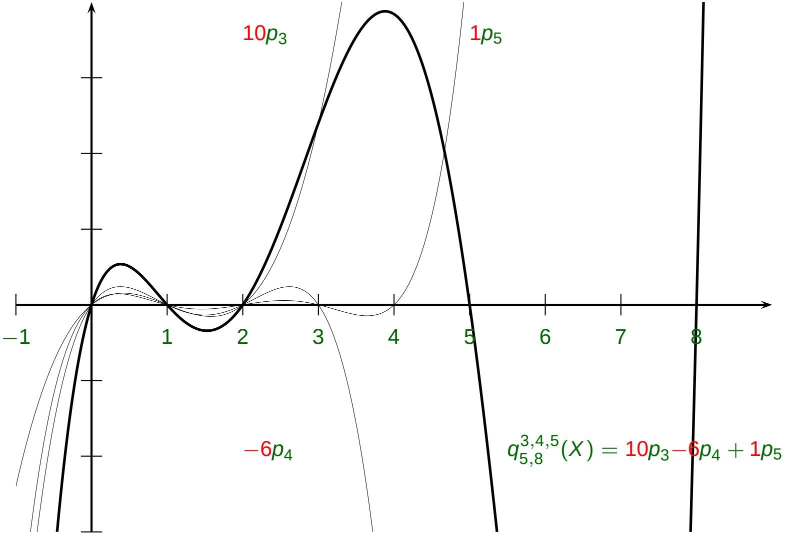
Example (Linear combination $q_{5,8}^{3,4,5}$ of p_3, p_4, p_5 with roots 5 and 8.)



Example (Linear combination $q^{3,4,5}_{5,8}$ of p_3, p_4, p_5 with roots 5 and 8.)



Example (Linear combination $q_{5,8}^{3,4,5}$ of p_3, p_4, p_5 with roots 5 and 8.)



Theorem

Let $0 \leq c_0 < \cdots < c_n \in \mathbb{N}$,
$1 \leq z_1 < \cdots < z_n \in \mathbb{N}$
s.t. $c_i \leq z_i \ \forall i$

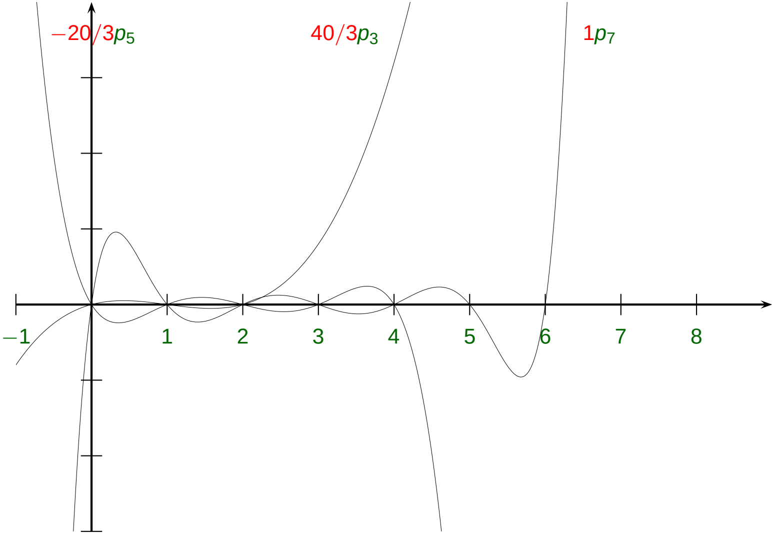
then $\exists!\ \alpha_0, \ldots, \alpha_n \in \mathbb{Q}$ *with* $\alpha_n := 1$ *s.t. for*

$$q_{(z_1,\ldots,z_n)}^{(c_0,\ldots,c_n)}(X) := \sum_{i=0}^{n} \alpha_i \left(\prod_{k=0}^{c_i - 1} (X - k) \right)$$

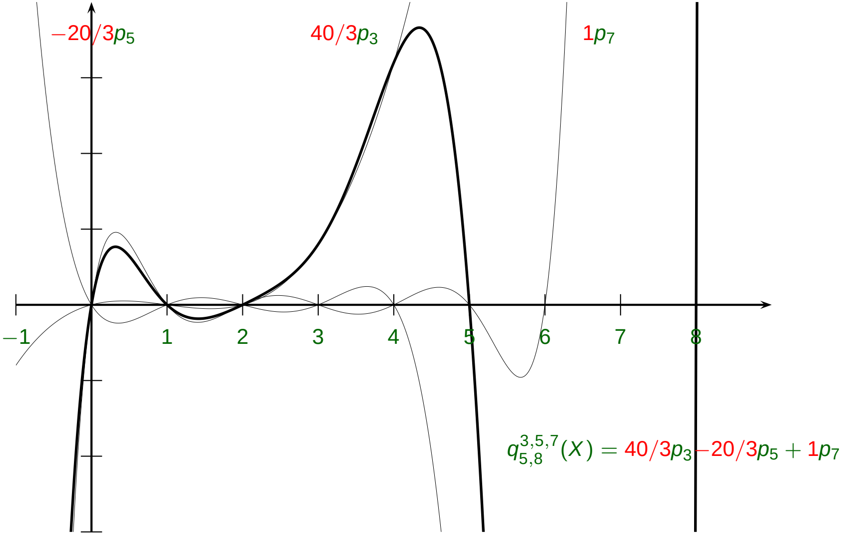
holds $q(z_i) = 0 \ \forall i$;

furthermore $\operatorname{sign}(\alpha_i \alpha_{i+1}) = -1 \ \forall i$,
the algebraic multiplicities of the roots z_i *are one,*
q *admits no other real roots* $\geq c_0$ *besides the* z_i,
calculation of α_i *is* $O(n^5 \log^4 n)$.

Example (Linear combination $q_{5,8}^{3,5,7}$ of p_3, p_5, p_7 with roots 5 and 8.)



Example (Linear combination $q_{5,8}^{3,5,7}$ of p_3, p_5, p_7 with roots 5 and 8.)



Example (Writing $q_{5,8}^{3,5,7}$ as linear combination of $q_{5,8}^{3,4,5}, q_{5,8}^{4,5,6}, q_8^{6,7}$ with positive coefficients.)

$$
\begin{array}{lccccccccc}
I & q_{5,8}^{3,4,5} & & & = & 10p_3 & -6p_4 & +1p_5 & & \\
II & & q_{5,8}^{4,5,6} & & = & & 4p_4 & -4p_5 & +1p_6 & \\
III & & & q_8^{6,7} & = & & & & -2p_6 & +1p_7 \\
IV & 2/3q_{5,8}^{3,4,5} & +q_{5,8}^{4,5,6} & & = & 20/3p_3 & & -10/3p_5 & +1p_6 & \\
V & 2(2/3q_{5,8}^{3,4,5} & +q_{5,8}^{4,5,6}) & +q_8^{6,7} & = & 40/3p_3 & & -20/3p_5 & & +1p_7
\end{array}
$$

$$
\begin{array}{llllllll}
 & & & + & + & + & - & + \\
- & f(b) & = f(a) & + f'(a)(b-a) & + \frac{f''(a)}{2!}(b-a)^2 & + \frac{f'''(a)}{3!}(b-a)^3 & + \frac{f''''(a)}{4!}(b-a)^4 & \\
+ & f'(b)(b-a) & = & + f'(a)(b-a) & + f''(a)(b-a)^2 & + \frac{f'''(a)}{2!}(b-a)^3 & + \frac{f''''(a)}{3!}(b-a)^4 & \\
- & f''(b)(b-a)^2 & = & & + f''(a)(b-a)^2 & + f'''(a)(b-a)^3 & + \frac{f''''(a)}{2!}(b-a)^4 & \\
- & f'''(b)(b-a)^3 & = & & & f'''(a)(b-a)^3 & + f''''(a)(b-a)^4 & \\
+ & & & & & & &
\end{array}
$$

$(+ + + - +)$ corresponds to $(f(a), f'(a), f''(a), f'''(a), f''''(a))$

$(- + - - +)$ corresponds to $(f(b), f'(b), f''(b), f'''(b), f''''(b))$ with $(a < b)$

$$\begin{array}{lllllll}
 & & + & + & + & - & + \\
- & \alpha f(b) = & \alpha f(a) & +\alpha f'(a)(b-a) & +\alpha\frac{f''(a)}{2!}(b-a)^2 & +\alpha\frac{f'''(a)}{3!}(b-a)^3 & +\alpha\frac{f''''(a)}{4!}(b-a)^4 \\
+ & \beta f'(b)(b-a) = & & +\beta f'(a)(b-a) & +\beta f''(a)(b-a)^2 & +\beta\frac{f'''(a)}{2!}(b-a)^3 & +\beta\frac{f''''(a)}{3!}(b-a)^4 \\
- & \gamma f''(b)(b-a)^2 = & & & +\gamma f''(a)(b-a)^2 & +\gamma f'''(a)(b-a)^3 & +\gamma\frac{f''''(a)}{2!}(b-a)^4 \\
- & \delta f'''(b)(b-a)^3 = & & & & \delta f'''(a)(b-a)^3 & +\delta f''''(a)(b-a)^4 \\
+ & & & & & & \\
\hline
\end{array}$$

$(+ + + - +)$ corresponds to $(f(a), f'(a), f''(a), f'''(a), f''''(a))$
$(- + - - +)$ corresponds to $(f(b), f'(b), f''(b), f'''(b), f''''(b))$ with $(a < b)$

$$\begin{array}{lcl}
 & & \quad + \qquad\qquad + \qquad\qquad + \qquad\qquad - \qquad\qquad + \\
- \quad 12f(b) & = & 12f(a) + 12f'(a)(b-a) + 12\frac{f''(a)}{2!}(b-a)^2 + 12\frac{f'''(a)}{3!}(b-a)^3 + 12\frac{f''''(a)}{4!}(b-a)^4 \\
+ -6f'(b)(b-a) & = & -\ 6f'(a)(b-a) - 6f''(a)(b-a)^2 - 6\frac{f'''(a)}{2!}(b-a)^3 - 6\frac{f''''(a)}{3!}(b-a)^4 \\
- \ 0f''(b)(b-a)^2 & = & +\ 0f''(a)(b-a)^2 + 0f'''(a)(b-a)^3 + 0\frac{f''''(a)}{2!}(b-a)^4 \\
- \ 1f'''(b)(b-a)^3 & = & 1f'''(a)(b-a)^3 + 1f''''(a)(b-a)^4 \\
+ & & \\
\hline
\end{array}$$

$(+ + + - +)$ corresponds to $(f(a), f'(a), f''(a), f'''(a), f''''(a))$
$(- + - - +)$ corresponds to $(f(b), f'(b), f''(b), f'''(b), f''''(b))$ with $(a < b)$

$$
\begin{array}{llcccccc}
 & & & + & + & + & - & + \\
- & 12f(b) & = & 12f(a) + 12f'(a)(b-a) & + 12\frac{f''(a)}{2!}(b-a)^2 & + 12\frac{f'''(a)}{3!}(b-a)^3 & + 12\frac{f''''(a)}{4!}(b-a)^4 \\
+ & -6f'(b)(b-a) & = & & -6f'(a)(b-a) - 6f''(a)(b-a)^2 & - 6\frac{f'''(a)}{2!}(b-a)^3 & - 6\frac{f''''(a)}{3!}(b-a)^4 \\
- & 0f''(b)(b-a)^2 & = & & + 0f''(a)(b-a)^2 & + 0f'''(a)(b-a)^3 & + 0\frac{f''''(a)}{2!}(b-a)^4 \\
- & 1f'''(b)(b-a)^3 & = & & & 1f'''(a)(b-a)^3 & + 1f''''(a)(b-a)^4 \\
+ & & & & & & \\
\hline
 & \multicolumn{7}{l}{12f(b) - 6f'(b)(b-a) + 1f'''(b)(b-a)^3} \\
 & - & - & - & & & \\
 & & = & 12f(a) + 6f'(a)(b-a) & + 0f''(a)(b-a)^2 & + 0f'''(a)(b-a)^3 & + \frac{1}{2}f''''(a)(b-a)^4 \\
 & & & + & + & & +
\end{array}
$$

$(+ + + - +)$ corresponds to $(f(a), f'(a), f''(a), f'''(a), f''''(a))$
$(- + - - +)$ corresponds to $(f(b), f'(b), f''(b), f'''(b), f''''(b))$ with $(a < b)$

$$+ \quad + \quad + \quad - \quad +$$

$$- \quad = 12f(a) + 12f'(a)(b-a) + 12\frac{f''(a)}{2!}(b-a)^2 + 12\frac{f'''(a)}{3!}(b-a)^3 + 12\frac{f''''(a)}{4!}(b-a)^4$$

$$+ \quad = - 6f'(a)(b-a) - 6f''(a)(b-a)^2 - 6\frac{f'''(a)}{2!}(b-a)^3 - 6\frac{f''''(a)}{3!}(b-a)^4$$

$$- \quad = + 0f''(a)(b-a)^2 + 0f'''(a)(b-a)^3 + 0\frac{f''''(a)}{2!}(b-a)^4$$

$$- \quad = 1f'''(a)(b-a)^3 + 1f''''(a)(b-a)^4$$

$$+$$

$$= 12f(a) + 6f'(a)(b-a) + 0f''(a)(b-a)^2 + 0f'''(a)(b-a)^3 + \frac{1}{2}f''''(a)(b-a)^4$$

$$+ \quad + \quad +$$

$(+ + + - +)$ corresponds to $(f(a), f'(a), f''(a), f'''(a), f''''(a))$
$(- + - - +)$ corresponds to $(f(b), f'(b), f''(b), f'''(b), f''''(b))$ with $(a < b)$

		+	+	+	−	+
−	$=$ 12	12	$12\frac{}{2!}$	$12\frac{}{3!}$	$12\frac{}{4!}$	
+	$=$	− 6	− 6	$-6\frac{}{2!}$	$-6\frac{}{3!}$	
−	$=$		0	0	$0\frac{}{2!}$	
−	$=$			1	1	
+						

$=$ 12	6	0	0	$\frac{1}{2}$	
	+	+			+

$(+ + + - +)$ corresponds to $(f(a), f'(a), f''(a), f'''(a), f''''(a))$
$(- + - - +)$ corresponds to $(f(b), f'(b), f''(b), f'''(b), f''''(b))$ with $(a < b)$

		$+$	$+$	$+$	$-$	$+$
$-$	$= 12$	12	$12\frac{}{2!}$	$12\frac{}{3!}$		$12 \cdot 1$
$+$	$=$	$-\ 6$	$-\ 6$	$-\ 6\frac{}{2!}$	$-$	$6 \cdot 4$
$-$	$=$		0	0		$0 \cdot 3 \cdot 4$
$-$	$=$			1		$1 \cdot 2 \cdot 3 \cdot 4$
$+$						

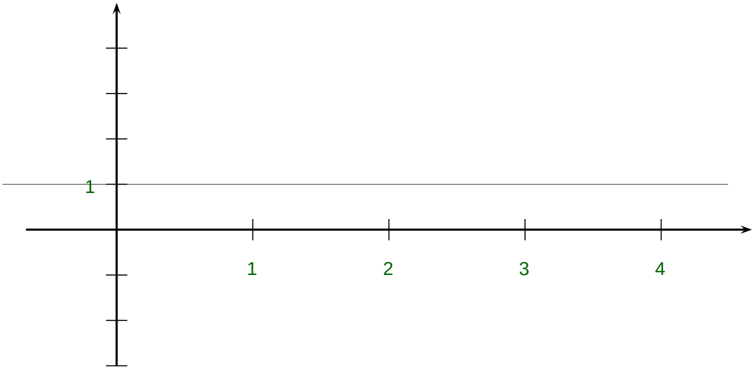
$= 12$	6	0	0		$\frac{1}{2} \cdot 4!$
	$+$	$+$			$+$

$(+ + + - +)$ corresponds to $(f(a), f'(a), f''(a), f'''(a), f''''(a))$
$(- + - - +)$ corresponds to $(f(b), f'(b), f''(b), f'''(b), f''''(b))$ with $(a < b)$

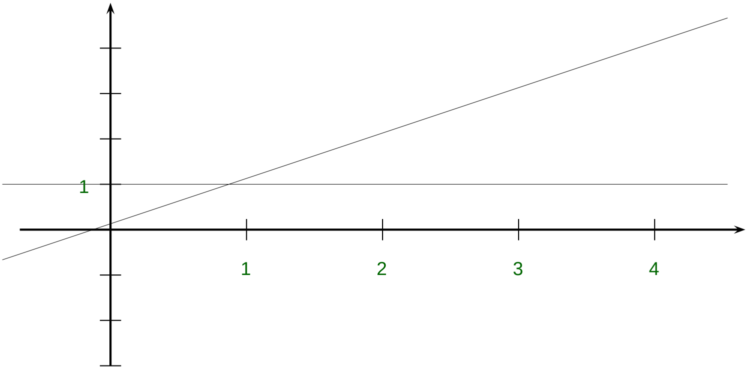
		$+$	$+$	$+$	$-$	$+$
$-$	$=$	$12 \cdot 1$	$12 \cdot 1$	$12 \cdot 1$	$12 \cdot 1$	$12 \cdot 1$
$+$	$=$		$-\ 6 \cdot 1$	$-\ 6 \cdot 2$	$-\ 6 \cdot 3$	$-\ 6 \cdot 4$
$-$	$=$			$0 \cdot 1 \cdot 2$	$0 \cdot 2 \cdot 3$	$0 \cdot 3 \cdot 4$
$-$	$=$				$1 \cdot 1 \cdot 2 \cdot 3$	$1 \cdot 2 \cdot 3 \cdot 4$
$+$						
	$=$	$12 \cdot 0!$	$6 \cdot 1!$	$0 \cdot 2!$	$0 \cdot 3!$	$\frac{1}{2} \cdot 4!$
		$+$	$+$			$+$

$(+ + + - +)$ corresponds to $(f(a), f'(a), f''(a), f'''(a), f''''(a))$
$(- + - - +)$ corresponds to $(f(b), f'(b), f''(b), f'''(b), f''''(b))$ with $(a < b)$

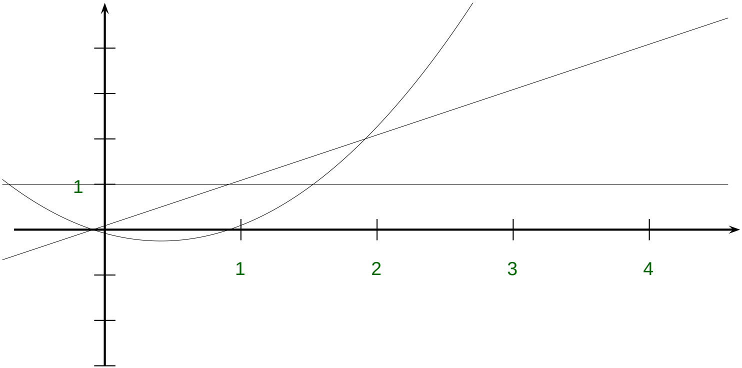
		$+$	$+$	$+$	$-$	$+$		
$-$	$=$	$12p_0(0)$	$12p_0(1)$	$12p_0(2)$	$12p_0(3)$	$12p_0(4)$	$p_0(X) :=$	1
$+$	$=$		$-\ 6 \cdot 1$	$-\ 6 \cdot 2$	$-\ 6 \cdot 3$	$-\ 6 \cdot 4$		
$-$	$=$			$0 \cdot 1 \cdot 2$	$0 \cdot 2 \cdot 3$	$0 \cdot 3 \cdot 4$		
$-$	$=$				$1 \cdot 1 \cdot 2 \cdot 3$	$1 \cdot 2 \cdot 3 \cdot 4$		
$+$								



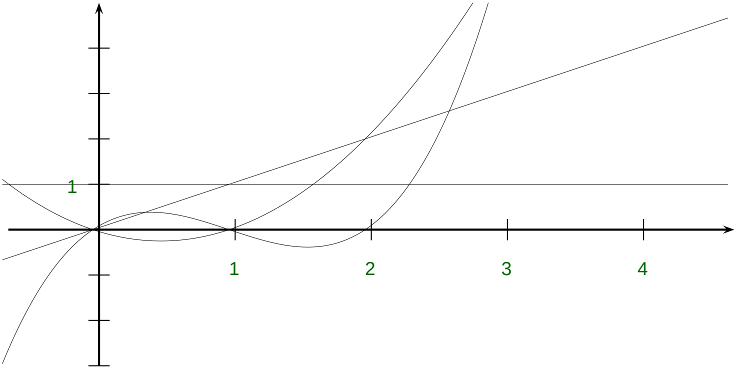
		+	+	+	−	+		
−	$=12p_0(0)$	$12p_0(1)$	$12p_0(2)$	$12p_0(3)$	$12p_0(4)$		$p_0(X) :=$	1
+	$=$		$-\ 6p_1(1)$	$-\ 6p_1(2)$	$-\ 6p_1(3)$	$-\ 6p_1(4)$	$p_1(X) :=$	X
−	$=$			$0 \cdot 1 \cdot 2$	$0 \cdot 2 \cdot 3$	$0 \cdot 3 \cdot 4$		
−	$=$				$1 \cdot 1 \cdot 2 \cdot 3$	$1 \cdot 2 \cdot 3 \cdot 4$		
+								



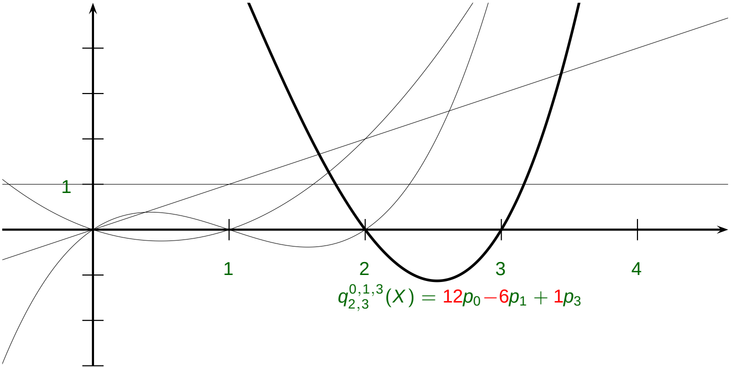
		$+$	$+$	$+$	$-$	$+$		
$-$	$=$	$12p_0(0)$	$12p_0(1)$	$12p_0(2)$	$12p_0(3)$	$12p_0(4)$	$p_0(X) :=$	1
$+$	$=$		$-\ 6p_1(1)$	$-\ 6p_1(2)$	$-\ 6p_1(3)$	$-\ 6p_1(4)$	$p_1(X) :=$	X
$-$	$=$			$0p_2(2)$	$0p_2(3)$	$0p_2(4)$	$p_2(X) :=$	$X(X-1)$
$-$	$=$				$1 \cdot 1 \cdot 2 \cdot 3$	$1 \cdot 2 \cdot 3 \cdot 4$		
$+$								



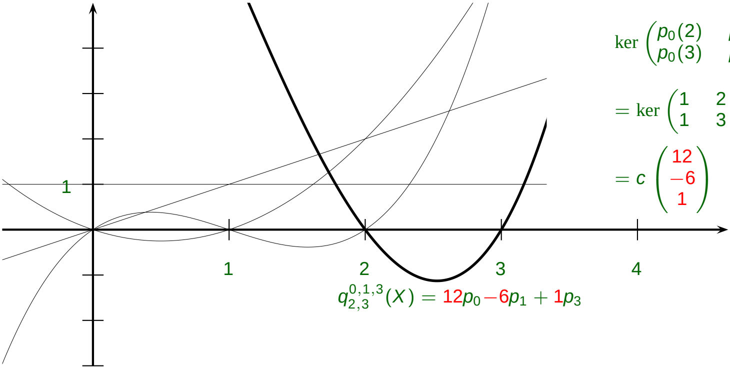
$$
\begin{array}{cccccccc}
 & & + & + & + & - & + & \\
- & = 12p_0(0) & 12p_0(1) & 12p_0(2) & 12p_0(3) & 12p_0(4) & p_0(X) := & 1 \\
+ & = & -\ 6p_1(1) & -\ 6p_1(2) & -\ 6p_1(3) & -\ 6p_1(4) & p_1(X) := & X \\
- & = & & 0p_2(2) & 0p_2(3) & 0p_2(4) & p_2(X) := & X(X-1) \\
- & = & & & 1p_3(3) & 1p_3(4) & p_3(X) := & X(X-1)(X-2) \\
+ & & & & & & &
\end{array}
$$



		$+$	$+$	$+$	$-$	$+$	
$-$	$=$	$12p_0(0)$	$12p_0(1)$	$12p_0(2)$	$12p_0(3)$	$12p_0(4)$	$p_0(X) := 1$
$+$	$=$		$-\ 6p_1(1)$	$-\ 6p_1(2)$	$-\ 6p_1(3)$	$-\ 6p_1(4)$	$p_1(X) := X$
$-$	$=$			$0p_2(2)$	$0p_2(3)$	$0p_2(4)$	$p_2(X) := X(X-1)$
$-$	$=$				$1p_3(3)$	$1p_3(4)$	$p_3(X) := X(X-1)(X-2)$
$+$							



$$
\begin{array}{lllllll}
 & + & + & + & - & + & \\
- \; = & 12p_0(0) & 12p_0(1) & 12p_0(2) & 12p_0(3) & 12p_0(4) & p_0(X) := 1 \\
+ \; = & & -\;6p_1(1) & -\;6p_1(2) & -\;6p_1(3) & -\;6p_1(4) & p_1(X) := X \\
- \; = & & & 0p_2(2) & 0p_2(3) & 0p_2(4) & p_2(X) := X(X-1) \\
- \; = & & & & 1p_3(3) & 1p_3(4) & p_3(X) := X(X-1)(X-2) \\
+ & & & & & &
\end{array}
$$



$$q_{2,3}^{0,1,3}(X) = 12p_0 - 6p_1 + 1p_3$$

$$
\ker \begin{pmatrix} p_0(2) & p_1(2) & p_3(2) \\ p_0(3) & p_1(3) & p_3(3) \end{pmatrix}
= \ker \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 6 \end{pmatrix}
= c \begin{pmatrix} 12 \\ -6 \\ 1 \end{pmatrix} \quad (c \in \mathbb{Q})
$$