Constructive Homological Algebra and Applications

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1 Introduction.

Standard homological algebra is not *constructive*, and this is frequently the source of serious problems when *algorithms* are looked for. In particular the usual exact and spectral sequences of homological algebra frequently are in general not sufficient to obtain some unknown homology or homotopy group. We will explain it is not difficult to fill in this gap, the main tools being on one hand, from a mathematical point of view, the so-called Homological Perturbation Lemma, and on the other hand, from a computational point of view, Functional Programming.

We will illustrate this area of constructive mathematics by applications in two domains:

- Commutative Algebra frequently meets homological objects, in particular when resolutions are involved (syzygies). Constructive Homological Algebra produces new methods to process old problems such as homology of Koszul complexes, resolutions, minimal resolutions. The solutions so obtained are *constructive* and therefore more complete thant the usual ones, an important point for their concrete use.
- Algebraic Topology is the historical origin of Homological Algebra. The usual exact and spectral sequences of Algebraic Topology can be easily transformed into new effective versions, giving algorithms computing for example unknown homology and homotopy groups in wide standard contexts. In particular the effective version of the Eilenberg-Moore spectral sequence gives a very simple solution for the old Adams' problem: what algorithm could compute the homology groups of iterated loop spaces?

Thanks are due to Ana Romero who carefully proofread several sections of this text.

2 Standard Homological Algebra.

We briefly recall in this section the minimal standard background of homological algebra. We mainly concentrate on definitions and basic results. Many good textbooks are available for the corresponding proofs, the main one being maybe [36]. The only problem almost never considered in these books is the relevant *computability problem*. Besides giving the expected background, our aim consists in making obvious why standard homological algebra does not at all satisfy the modern *constructiveness requirement*.

2.1 Ingredients.

Homological algebra is a general style of cooking where the main ingredients are a *ground ring* $\mathfrak{R}$, *chain-complexes*, *chain groups*, *boundary maps*, *chains*, *boundaries*, *cycles*, *homology classes*, *homology groups*, *exact sequences* and, the last but not the least, *spectral sequences*. In particular we do not consider here the cohomological operations, where a good reference is [46]; this roughly defines the frontier between which is covered in this text and which is not. Let us remark also that cohomological operations would probably be filed by most algebraic topologists in Algebraic *Topology*, but we will explain later [redacted] why such a discussion in fact does not make sense. In the same way, modern homological algebra requires the notion of algebraic operad [38], a completely different approach toward constructive algebraic topology, very interesting, but which unfortunately did not yet produce significant concrete computer programs. An operad is nothing but an algebra of generalized *abstract* cohomological operations.

Homological algebra was invented to systematically organize the algebraic environment needed by the computation of the homology groups associated with some topological objects. The first systematic presentation of Algebraic Topology heavily based on homological algebra certainly is [22], another convenient reference for a detailed presentation and the relevant proofs of most elementary facts. Now homological algebra is a fundamental tool in many domains not directly connected to algebraic topology. Section 5 here devoted to the so-called *Spencer cohomology*, where homological algebra is applied to commutative algebra and local non-linear PDE systems, is a typical example.

2.2 Chain-complexes.

2.2.1 Definitions.

The *ground ring* $\mathfrak{R}$ is an arbitrary unitary commutative ring; in the topological case, an abelian *group*, without any multiplicative structure can also be considered, frequent when studying spectral sequences, because of "coefficients" that are other homology groups. In algebraic topology, $\mathfrak{R}$ is often $\mathbb{Z}$, the most general case because of the *universal coefficient theorem* [36, V.11]: if you know the homology groups with respect to the ground ring $\mathbb{Z}$, you can easily deduce the same homology groups with respect to any other ground ring of coefficients. But because of the power of the $\mathbb{Z}$-homology groups, they are of course the most difficult to be computed. Other less ambitious possibilities are $\mathfrak{R} = \mathbb{Q}$ or $\mathbb{Z}_p$ (p being a prime number); note that in algebraic topology, $\mathbb{Z}_p$ does not denote the p-adic ring, it is simply $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$; the rings $\mathbb{Q}$ and $\mathbb{Z}_p$ are in fact fields, making easier certain calculations, and the last but not obvious step then consists in reconstructing the $\mathbb{Z}$-homology groups from the $\mathbb{Q}$ and $\mathbb{Z}_p$ homology groups, the main tool being the Bockstein-Browder spectral sequence [42, Chap.10]; a critical and interesting open problem consists in obtaining a *constructive* version of this spectral sequence.

UOStated 1 [1]— *In these notes, unless otherwise stated, some underlying ring $\mathfrak{R}$ is assumed given. A module is therefore implicitly an $\mathfrak{R}$-module.*

In algebraic topology, the most useful ring is $\mathbb{Z}$ and you can assume this convenient hypothesis. In commutative algebra, the ground ring will be most often a field; a module is then a vector space, making some problems significantly easier; but this apparent comfort is also misleading: *effective* homology is as useful in commutative algebra as in algebraic topology.

Definition 2 — A *chain-complex* C_* is a pair of sequences $C_* = (C_q, d_q)_{q\in\mathbb{Z}}$ where:

- For every $q \in \mathbb{Z}$, the component C_q is an $\mathfrak{R}$-module, the *chain group* of degree q.
- For every $q \in \mathbb{Z}$, the component d_q is a module morphism $d_q : C_q \to C_{q-1}$, the differential map.
- For every $q \in \mathbb{Z}$, the composition $d_q d_{q+1}$ is null: $d_q d_{q+1} = 0$.

$$\begin{tikzcd}
\cdots & C_{q-2} \arrow[l, "d_{q-2}"'] & C_{q-1} \arrow[l, "d_{q-1}"'] \arrow[ll, bend left, "0"] & C_q \arrow[l, "d_q"'] \arrow[ll, bend left, "0"] & C_{q+1} \arrow[l, "d_{q+1}"'] \arrow[ll, bend left, "0"] & C_{q+2} \arrow[l, "d_{q+2}"'] \arrow[ll, bend left, "0"] & \cdots \arrow[l, "d_{q+3}"'] \arrow[ll, bend left, "0"]
\end{tikzcd}$$

Definition 3 — If $C_* = (C_q, d_q)_{q\in\mathbb{Z}}$ is a chain-complex, the module C_q is called the *chain group* of degree q (in fact it is a module, but the terminology *chain group* is so traditional...), or the group of *q-chains*. The image $B_q = d_{q+1}(C_{q+1}) \subset C_q$ is the (sub) group of *q-boundaries*. The kernel $Z_q = \ker(d_q) \subset C_q$ is the group of *q-cycles*. The relation $d_q \circ d_{q+1} = 0$ is equivalent to the inclusion relation $B_q \subset Z_q$: every boundary is a cycle but the converse in general is not true. The "difference" (quotient) $H_q = Z_q/B_q$ is the *homology group* $H_q(C_*)$, again in fact a module.

Another possible point of view consists in considering $C_* = \oplus_q C_q$ is a *graded module* and the differential $d : C_* \to C_{*-1}$ is a graded morphism of degree -1

[1] Unless otherwise stated.

satisfying $d^2 = 0$. According to the situation one point of view or other is more convenient, and you must be able to immediately translate from one point of view to the other one.

Most often, the chain groups in negative degree are null: $q < 0 \Rightarrow C_q = 0$, so that it becomes tempting to decide to index by $q \in \mathbb{N}$ instead of $q \in \mathbb{Z}$, but experience shows it is not a good idea. The main reason is that this requires specific definitions in degree 0, the cycle group being then no longer defined, unless you decide to put an extra $C_{-1} = 0$ and the problem is transferred at -1… In particular when we write down corresponding programs, a choice $q \in \mathbb{N}$ would require specific code for the particular case $q = 0$, which quickly becomes painful, without any advantage.

Definition 4 — More generally, let C_* be a chain-complex and M a *coefficient group*, that is, an $\mathfrak{R}$-module. Then C_* and M generate two other chain-complexes:

- $C_* \otimes_{\mathfrak{R}} M := (C_q \otimes_{\mathfrak{R}} M, d_q \otimes_{\mathfrak{R}} \mathrm{id}_M)$. The corresponding cycles, boundaries and homology groups are then usually denoted by $Z_q(C_*; M)$, $B_q(C_*; M)$ and $H_q(C_*; M)$. We speak then of homology groups "with coefficients in M".
- $\mathrm{Hom}(C_*; M) := (\mathrm{Hom}(C_q, M), d^q)$ with d^q the morphism $d^q : \mathrm{Hom}(C_q, M) \to \mathrm{Hom}(C_{q+1}, M)$ *dual* to d_{q+1}. The corresponding objects are then denoted with q-*exponents*: $Z^q(C_*; M)$, $B^q(C_*; M)$ and $H^q(C_*; M)$. In this case, when the differential has degree $+1$, it is common to call the complex a *co*chain-complex, to call the corresponding objects *co*cycles, *co*boundaries, *co*homology groups (not homology cogroups!).

Others prefere to reverse the indices, deciding that $C^q(C_*; M) := \mathrm{Hom}(C_{-q}, M)$; question of taste. the cohomological context will be rarely considered in these notes.

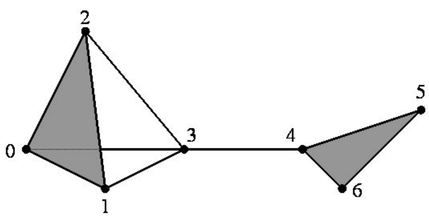
2.2.2 Simplicial complexes.

Definition 5 — A *simplicial complex* K is a pair $K = (V, S)$ where:

- The component V is a totally ordered[2] set, the set of vertices of K.
- The component S is a set of *non-empty finite* parts of V, the *simplices* of K, satisfying the properties:
  - For every $v \in V$, the singleton $(v) \in S$.
  - For every $\sigma \in V$, then $\emptyset \neq \sigma' \subset \sigma$ implies $\sigma' \in V$.

For example the small simplicial complex drawn here:

[2] A more intrinsic definition does not require such an order, but the associated chain-complex is significantly bigger; it can always be reduced [redacted] over a much smaller chain-complex, the definition of which requires a total order over the vertex set.



The *butterfly* simplicial complex
(Yvon Siret's terminology).

is mathematically defined as the object $B = (V, S)$ with:

$$V = (0, 1, 2, 3, 4, 5, 6)$$

$$S = \left\{ \begin{array}{l} (0), (1), (2), (3), (4), (5), (6), \\ (0,1), (0,2), (0,3), (1,2), (1,3), (2,3), (3,4), (4,5), (4,6), (5,6), \\ (0,1,2), (4,5,6) \end{array} \right\}$$

In other words, the second component, the simplex list, gives the list of all vertex combinations which are (abstractly) spanned by a simplex. The vertex set V could be for example ordered as the integers are. Note also, because the vertex set is ordered, the list of vertices of a simplex is also ordered, which allows us to use a *sequence* notation $(\cdots)$ and not a subset notation $\{\cdots\}$ for a simplex and also for the total vertex list V.

A simplicial complex can be infinite. For example if $V = \mathbb{N}$ and $S = \{(n)\}_{n\in\mathbb{N}} \cup \{(0, n)\}_{n\geq 1}$, the simplicial complex so obtained could be understood as an infinite bunch of segments. Standard algebraic topology proves that most "sensible" homotopy types can be modelled as simplicial complexes, often infinite. We will see the notion of simplicial set, roughly similar but more sophisticated, is also much more powerful to reach this goal[3].

2.2.3 From simplicial complexes to chain-complexes.

Definition 6 — Let $K = (V, S)$ be a simplicial complex. Then the set $S_n(K)$ of n-simplices of K is the set made of the simplices of cardinality $n + 1$.

For example the set of simplices $S_0(K)$ is the set of singletons $S_0(K) = \{(v)\}_{v\in V}$. The set of 2-simplices of the butterfly B is $\{(0, 1, 2), (4, 5, 6)\}$; in the same case, the set of 1-simplices has ten elements.

Definition 7 — Let $K = (V, S)$ be a simplicial complex. Then the *chain-complex* $C_*(K)$ *canonically associated with* K is defined as follows. The chain group $C_n(K)$

[3]There is here an amusing bug of terminology: the notion of simplicial set, due to Sam Eilenberg, is more *complex* than the notion of simplicial… complex.

is the free module generated by $S_n(K)$. Let $(v_0, \ldots, v_n)$ be an n-simplex, that is, a generator of $S_n(K)$. The boundary of this generator is then defined as:

$$d_n((v_0, \ldots, v_n)) = (v_1, v_2, \ldots, v_n) - (v_0, v_2, v_3, \ldots, v_n) + \cdots + (-1)^n (v_0, v_1, \ldots, v_{n-1})$$

and this definition is linearly extended to $C_n(K)$.

A variant of this definition is important.

Definition 8 — Let $K = (V, S)$ be a simplicial complex. Let $n \geq 1$ and $0 \leq i \leq n$ be two integers n and i. Then the *face operator* ∂_i^n is the linear map $\partial_i^n : C_n(K) \to C_{n-1}(K)$ defined by:

$$\partial_i^n((v_0, \ldots, v_n)) = (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n) \ :$$

the i-th vertex of the simplex is removed, so that an $(n-1)$-simplex is obtained.

Remark 9 — *The boundary operator d_n is the alternate sum:*

$$d_n := \sum_{i=0}^{n} (-1)^i \partial_i^n.$$

This definition will be generalized later [redacted] thanks to the notion of simplicial *set*.

Our butterfly example is then sufficient to understand the nature of the notions of chain, cycle, boundary and homology class. An example of 1-chain is $c = (1,3)+(3,4)+(4,5) \in C_1(B)$; we here have chosen an example as close as possible to the usual concrete notion of "chain", but $c' = (0,2)+(3,4)+(5,6)$ is a chain as well. The boundaries are $d_1(c) = -(1)+(5)$ and $d_1(c') = +(2)-(0)+(4)-(3)+(6)-(5)$. The chain $c_1 = (1,2) + (2,3) - (1,3)$ is a cycle, but is not a boundary, it is an "interesting" cycle, the homology class of which is non-null. On the contrary the cycle $c_2 = (4,5) + (5,6) - (4,6)$ is trivial, it is the boundary of the 2-chain $(4,5,6)$, and its homology class is null. If a cycle is homologous to 0, it can be in general the boundary of several *different* chains; for example, in our butterfly, the 0-cycle $(3) - (1)$ is the boundary of the 1-chain $(1,3)$, but also the boundary of $(1,2) + (2,3)$, a different 1-chain.

2.2.4 Computing homology groups.

Computing a homology group amounts to computing the relevant boundary matrices, and to determine a kernel, an image and the quotient of the first one by the second one. For example, if we want to compute the homology group $H_1(B)$, the 1-dimensional homology group of our butterfly, we have to describe the kernel of d_1:

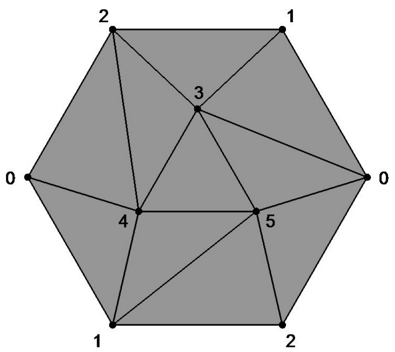
$$\begin{array}{rcr} \ker d_1 & = & \mathfrak{R}((0,1) + (1,2) - (0,2)) \\ & & \oplus\ \mathfrak{R}((0,1) + (1,3) - (0,3)) \\ & & \oplus\ \mathfrak{R}((0,2) + (2,3) - (0,3)) \\ & & \oplus\ \mathfrak{R}((4,5) + (5,6) - (4,6)) \end{array}$$

and the image of d_2:

$$\begin{aligned} \operatorname{im} d_2 \quad = \quad & \mathfrak{R}((0,1)+(1,2)-(0,2)) \\ & \oplus \mathfrak{R}((4,5)+(5,6)-(4,6)). \end{aligned}$$

Note in particular the tempting cycle $(1,2)+(2,3)-(1,3)$ is the alternate sum of the first three ones in the description of $\ker d_1$. So that the homology group $H_1(B)$ is *isomorphic* to $\mathfrak{R}^2$ with $(0,1)+(1,3)-(0,3)$ and $(0,2)+(2,3)-(0,3)$ as possible *representants* of generators, but adding to such a representant an arbitrary boundary gives another representant of the same homology class.

These computations quickly become complicated and it is then better – or necessary – to be helped by a computer. Let us examine for example the case of the real projective plane $P^2\mathbb{R}$. It can be proved the minimal triangulation of $P^2\mathbb{R}$ as a simplicial complex is described by this figure:



The projective plane is the quotient of the 2-sphere by the antipodal relation. Taking a hemisphere, that is, a disk, as a fundamental domain, we must then identify two opposite points on the limit circle. Replacing the disk by the homeomorphic hexagon, we obtain the figure above, the identification of opposite points of the perimeter explaining the *apparent* repetition of the vertices 0, 1 and 2 and the corresponding edges.

This simplicial complex has six vertices, fifteen edges and ten triangles. The 1-skeleton is a complete graph with six vertices: any two vertices are connected by an edge[4]. Computing by hand the homology groups of this simplicial complex is a little lengthy. The Kenzo program obtains the result as follows.

```
> (setf P2R
    (build-finite-ss
     '(v0 v1 v2 v3 v4 v5
       1 e01 (v1 v0) e02 (v2 v0) e03 (v3 v0) e04 (v4 v0) e05 (v5 v0)
         e12 (v2 v1) e13 (v3 v1) e14 (v4 v1) e15 (v5 v1) e23 (v3 v2)
         e24 (v4 v2) e25 (v5 v2) e34 (v4 v3) e35 (v5 v3) e45 (v5 v4)
       2 t013 (e13 e03 e01) t014 (e14 e04 e01) t024 (e24 e04 e02)
         t025 (e25 e05 e02) t035 (e35 e05 e03) t123 (e23 e13 e12)
         t125 (e25 e15 e12) t145 (e45 e15 e14) t234 (e34 e24 e23)
         t345 (e45 e35 e34)))) ✠
```

[4] A necessarily *unique* edge in the context of simplicial complexes.

```
Checking the 0-simplices...
Checking the 1-simplices...
Checking the 2-simplices...
[K1 Simplicial-Set]
```

A Kenzo listing of this sort must be read as follows. The initial '`>`' is the Lisp prompt of this implementation. The user types out a Lisp statement, here `(setf...e35 e34))))` and the maltese cross ✠ (in fact not visible on the user screen) marks here the end of the Lisp statement, the right number of closing parentheses is reached. The corresponding Return key asks Lisp to evaluate the statement. Here a finite simplicial *set* is constructed according to the given description, it is assigned to the symbol `P2R`, and *returned*, that is, displayed: it is the Kenzo object #1 (`K1`) and it is a simplicial set; this is just a small external display, the internal structure is not shown. Kenzo explains beforehand it verifies the coherence of the definition of the simplicial set.

This construction of the projective plane is a little laborious. In general the simplicial complexes are not used in "good" algebraic topology; we have in fact used the more general notion of simplicial set. The definition goes as follows: the `build-finite-ss` Kenzo function is used, which requires one argument, a list describing the finite simplicial set to be constructed; firstly the vertices are given (six symbols `v0` to `v5`, then the edges (fifteen symbols `e01` to `e45`) and for each of them their both "faces" (ends), and finally ten triangles and their faces (sides).

To arouse the interest for general simplicial sets, we immediately give the minimal combinatorial definition of the projective plane as a simplicial *set*:

```
> (setf short-P2R
    (build-finite-ss
     '(v 1 e (v v) 2 t (e v e)))) ✠
Checking the 0-simplices...
Checking the 1-simplices...
Checking the 2-simplices...
[K6 Simplicial-Set]
```

It is explained here only one vertex '`v`' is necessary, one edge '`e`' and one triangle '`t`'. Both ends of the edge are the unique vertex. The sides 0 and 2 of the triangle are the unique edge, and the side 1 is collapsed on the vertex. It is clear `P2R` and `short-P2R` are homeomorphic, and the second definition is much more natural, but the underlying theory is not so easy.

The boundary matrices of `P2R` are:

```
> (chcm-mat P2R 1) ✠
========== MATRIX 6 lines + 15 columns =====
L1=[C1=-1][C2=-1][C3=-1][C4=-1][C5=-1]
L2=[C1=1][C6=-1][C7=-1][C8=-1][C9=-1]
L3=[C2=1][C6=1][C10=-1][C11=-1][C12=-1]
L4=[C3=1][C7=1][C10=1][C13=-1][C14=-1]
```

```
L5=[C4=1][C8=1][C11=1][C13=1][C15=-1]
L6=[C5=1][C9=1][C12=1][C14=1][C15=1]
========== END-MATRIX
```

between degrees 1 and 0 and :

```
> (chcm-mat P2R 2) ✠
========== MATRIX 15 lines + 10 columns =====
L1=[C1=1][C2=1]
L2=[C3=1][C4=1]
L3=[C1=-1][C5=1]
L4=[C2=-1][C3=-1]
L5=[C4=-1][C5=-1]
L6=[C6=1][C7=1]
L7=[C1=1][C6=-1]
L8=[C2=1][C8=1]
L9=[C7=-1][C8=-1]
L10=[C6=1][C9=1]
L11=[C3=1][C9=-1]
L12=[C4=1][C7=1]
L13=[C9=1][C10=1]
L14=[C5=1][C10=-1]
L15=[C8=1][C10=1]
========== END-MATRIX
```

between degrees 2 and 1. Because large matrices can happen, a sparse display is given; for example, for the last matrix, the row (line) 1 has only two non null terms 1 in columns 1 and 2, the row 7 has a 1 in column 1 and a -1 in column 6, etc.

Computing the homology groups amounts to determining the kernel of the first matrix, the image of the second one and the quotient of the kernel by the image, a work a little painful. Significantly less painful for `short-P2R`:

```
> (chcm-mat short-P2R 1) ✠
========== MATRIX 1 lines + 1 columns =====
L1=
========== END-MATRIX
> (chcm-mat short-P2R 2) ✠
========== MATRIX 1 lines + 1 columns =====
L1=[C1=2]
========== END-MATRIX
```

which means the chain-complex of `short-P2R` is:

$$0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \leftarrow 0$$

if the ground ring is $\mathbb{Z}$ and it is then clear $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}_2$ and $H_2 = 0$.

The homology groups of `P2R` and `short-P2R` can be computed by Kenzo, for example the H_1 groups.

```
> (homology P2R 1) ✠
Homology in dimension 1 :
Component Z/2Z
---done---
> (homology short-P2R P2R 1) ✠
Homology in dimension 1 :
Component Z/2Z
---done---
```

The actual Kenzo display is more verbose and we keep here only the interesting parts. You can do the same in degrees 0 and 2, both spaces have the same homology groups, which "confirms" – but does not prove – both spaces are homeomorphic.

2.3 Chain-complex morphisms.

2.3.1 Definition.

Definition 10 — Let $A_* = \{A_q, d_q\}_q$ and $B_* = \{B_q, d_q\}_q$ be two chain-complexes[5]. A *chain-complex morphism* $f : A_* \to B_*$ is a collection of linear morphisms $f = \{f_q : A_q \to B_q\}_q$ satisfying the differential condition: for every q, the relation $f_{q-1}d_q = d_q f_q$, or more simply $df = fd$:

$$\begin{tikzcd} A_{q-1} \arrow[d, "f"'] & B_q \arrow[l, "d"'] \arrow[d, "f"] \\ B_{q-1} & B_q \arrow[l, "d"] \end{tikzcd}$$

is satisfied.

More and more frequently, we will not indicate the indices of morphisms, clearly implied by context. Also we use the same notation for a morphism and some other morphisms directly deduced from the first one.

If $f : A_* \to B_*$ is a chain-complex morphism, many other maps are naturally induced; most often they are denoted by the same symbol, f in this case. Because of the differential condition, the image of a cycle is a cycle and we have induced maps $f : Z_q(A_*) \to Z_q(B_*)$, the same for the boundaries $f : B_q(A_*) \to B_q(B_*)$, and for homology classes and homology groups $f : H_*(A_*) \to H_*(B_*)$.

2.3.2 Simplicial morphisms.

Definition 11 — Let $K = (V, S)$ *and* $K' = (V', S')$ *be two simplicial complexes. A (simplicial) morphism* $f : K \to K'$ *is a map* $f : V \to V'$ *satisfying the conditions:*

[5]We do not hesitate to use the same symbol, d in this case, for different… differentials, the context being sufficient to avoid any ambiguity.

- *The map f is compatible*[6] *with the orders defined over V and V', see Definition 5. More precisely, if $v \leq v'$ in V, then $f(v) \leq f(v')$ in V'*[7].
- *If $\sigma \in S$, then $f(\sigma) \in S'$.*

In other words, if $v_0 < \cdots < v_k$ span a simplex of K, then $f(v_0) \leq \cdots \leq f(v_k)$ span a simplex of V', but in the second sequence, repetitions are allowed.

Now a simplicial morphism $f : K \to K'$ induces a chain-complex morphism again denoted by $f : C_*(K) \to C_*(K')$. Only one possible definition. If $\sigma \in S_k(K)$ is a k-simplex of K, therefore a generator of $C_k(K)$, two cases; if $f(\sigma)$ again is a k-simplex of K', that is, if there is no repetition in the images of the vertices, then $f(\sigma) := \ldots f(\sigma)$ where the left hand side is understood in $C_k(K')$ and the right hand one in $S_k(K')$; if on the contrary $f(\sigma) \in S_{\boldsymbol{\ell}}(K')$ with $\ell < k$, then we decide $f(\sigma) := 0$ in $C_{\boldsymbol{k}}(K')$: the image simplex is "squeezed" — we will see later the appropriate terminology is "degenerate", see Definition 106 — and this simplex do not anymore contribute to homology. We advise the reader to verify the chain-complex *map* $f : C_*(K) \to C_*(K')$ so defined is compatible with the differentials, and therefore actually is a chain-complex *morphism*, the underlying sign game is instructive. Examples of simplicial morphisms will be soon used in the proof of Theorem 18.

2.4 Homotopy operators.

2.4.1 Definition and first properties.

Definition 12 — Let $A_* = \{A_q, d_q\}_q$ and $B_* = \{B_q, d_q\}_q$ be two chain-complexes. A homotopy operator $h : A_* \to B_*$ is a collection $h = \{h_q : A_q \to B_{q+1}\}_q$ of linear maps. In other words, it is a linear map $h : A_* \to B_{*+1}$ of degree $+1$, this degree being implicitly implied by the index '$* + 1$' of B_{*+1}.

In particular, no compatibility condition is required with the respective differentials of A_* and B_*. In the interesting cases, the homotopy operator is rather "seriously non-compatible" with these differentials.

Definition 13 — Let $f, g : A_* \to B_*$ be two chain-complex morphisms. A homotopy operator $h : A_* \to B_{*+1}$ is a *homotopy between f and g* if the relation $g - f = dh + hd$ is satisfied.

The next diagram shows there is a unique way to understand this relation when

[6] Again a more general definition is possible, without any order defined over V and V', see [redacted], but its use is then significantly more technical and this matter is not directly our matter. Yet it is again a matter of *reductions* [redacted] !

[7] In particular $v < v'$ and $f(v) = f(v')$ is possible.

you start from A_q and arrive at B_q:

$$\begin{tikzcd}
A_{q-1} \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dr, "h"] & A_q \arrow[l, "d"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dr, "h"] & A_{q+1} \arrow[l, "d"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\
B_{q-1} & B_q \arrow[l, "d"] & B_{q+1} \arrow[l, "d"]
\end{tikzcd}$$

Proposition 14 — *If two chain-complex morphisms $f, g : A_* \to B_*$ are* homotopic*, then the induced maps $f, g : H_*(A_*) \to H_*(B_*)$ are* equal*.*

Proof. Let h be a homotopy between f and g. If z is a q-cycle representing the homology class $\mathfrak{h} \in H_q(A_*)$, then the relation $gz - fz = dhz + hdz$ is satisfied; but z is a cycle and $hdz = 0$, so that $gz - fz = dhz$, which expresses the cycles fz and gz representing the homology classes $f\mathfrak{h}$ and $g\mathfrak{h}$ are homologous, their difference is a boundary; and therefore $f\mathfrak{h} = g\mathfrak{h}$. ■

Definition 15 — A *homology equivalence* between two chain-complexes A_* and B_* is a pair (f, g) of chain-complex morphisms $f : A_* \to B_*$ and $g : B_* \to A_*$ such that gf is homotopic to id_{A_*} and fg is homotopic to id_{B_*}.

The terminology is not well stabilized, many authors use rather *chain equivalence*, or *homotopy equivalence*. We feel more simple and clear our terminology. We can also say that $f : A_* \to B_*$ is a homology equivalence if there exists a *homological inverse* $g : B_* \to A_*$ such that the pair (f, g) satisfies the above definition.

Proposition 16 — *If $f : A_* \to B_*$ is a homology equivalence, then the induced maps $\{f_q : H_q(A_*) \to H_q(B_*)\}_q$ are isomorphisms.*

Proof. The maps gf and fg are respectively *homotopic* to id_{A_*} and id_{B_*}, so that the induced maps $gf : H_q(A_*) \to H_q(A_*)$ and $fg : H_q(B_*) \to H_q(B_*)$ are *equal* to the corresponding identities. ■

2.4.2 Example.

Definition 17 — The *standard n-simplex* Δ^n of dimension n is the simplicial complex $(\underline{n}, \mathcal{P}_*(\underline{n}))$ where $\underline{n}$ is the set of integers $\underline{n} = (0, \ldots, n)$ from 0 to n and $\mathcal{P}_*(\underline{n})$ is the set of *non-empty* subsets of $\underline{n}$.

Theorem 18 — *The homology groups of the standard simplex Δ^n are null except $H_0(\Delta^n) = \mathfrak{R}$, the ground ring.*

Proof. The result is obvious when $n = 0$. Otherwise we can consider two simplicial morphisms $f : \Delta^0 \to \Delta^n$ and $g : \Delta^n \to \Delta^0$ where f(0) = 0 and $g(i) = 0$ for every i. The composition gf is the identity, the composition fg is not, but the induced map $fg : C_*(\Delta^n) \to C_*(\Delta^n)$ is homotopic to the identity. The needed homotopy operator $h : C_*(\Delta^n) \to C_{*+1}(\Delta^n)$ is defined as follows; let $\sigma = (i_0, \ldots, i_k)$ a k-simplex generator of $C_k(\Delta^n)$, that is, an ordered sequence of $k+1$ integers $i_0 < \cdots < i_k$ of $\underline{n}$. If $i_0 > 0$, we decide $h(\sigma) = (0, i_0, \ldots, i_k)$; if on the contrary $i_0 = 0$, then we decide $h(\sigma) = 0$. An interesting but elementary computation then shows $dh + hd = \mathrm{id}_{C_*(\Delta^n)} - fg$. So that the map $fg : H_*(\Delta^n) \to H_*(\Delta^n)$ is simply *equal* to the identity and $f : H_*(\Delta^0) \to H_*(\Delta^n)$ is an isomorphism. ∎

2.5 Exact sequences.

Definition 19 — A chain-complex $C_* = \{C_q, d_q\}_{q \in \mathbb{Z}}$ is *exact at degree* q if $\ker d_q = \operatorname{im} d_{q+1}$, in other words if $H_q(C_*) = 0$, or if $Z_q(C_*) = B_q(C_*)$: every q-cycle is a q-boundary, no "interesting" cycle in degree q. The chain-complex is *exact* if it is exact in every degree. In the same case, it is frequent also to state the chain-complex is *acyclic*; this does not mean there is no cycle, you must understand there is no *non-trivial* cycle, that is, a cycle which is not a boundary. The expressions "exact chain-complex", "acyclic chain-complex", "exact sequence" are perfectly synonymous.

Proposition 20 — *Let* (C_*, d) *be a chain-complex. If there exists a homotopy operator* $h : C_* \to C_{*+1}$ *satisfying* $\mathrm{id} = dh + hd$*, then the chain-complex* (C_*, d) *is acyclic (or exact).*

Proof. We can rewrite our relation $\mathrm{id} - 0 = dh + hd$, that is, the identity map is homotopic to the null map. The induced maps in homology therefore are *equal*. These induced maps are respectively the identity maps and the null maps $H_*(C_*) \to H_*(C_*)$. If M is a module and if $\mathrm{id}_M = 0_M$, this is possible only if $M = 0$. ∎

Definition 21 — *A* short exact sequence *is a sequence of modules:*

$$0 \leftarrow C'' \stackrel{j}{\leftarrow} C \stackrel{i}{\leftarrow} C' \leftarrow 0$$

which is exact, that is in this case, the map i *is injective, the map* j *is surjective and* $\operatorname{im} i = \ker j$.

In particular the module C' is then canonically isomorphic to the kernel of j and C'' to the cokernel of i. One says the central module C is an *extension* of C'' *by* C'. In general there are several possible extensions. For example if the ground ring is $\mathbb{Z}$, there are two extensions of $\mathbb{Z}_6$ by $\mathbb{Z}_2$, namely $\mathbb{Z}_2 \oplus \mathbb{Z}_6$ and $\mathbb{Z}_{12}$ which are not isomorphic. The so-called *extension problem*, how to determine in a particular case which is the right extension when the left hand and right hand modules are known, is often a major problem in homological algebra.

If a "long" sequence C_* is exact, there is no reason the short sequence:

$$0 \leftarrow C_{q-1} \stackrel{d_q}{\leftarrow} C_q \stackrel{d_{q+1}}{\leftarrow} C_{q+1} \leftarrow 0$$

is exact. To make it exact we must force d_{q+1} to be injective and d_q to be surjective, and we obtain the short exact sequence:

$$0 \leftarrow \operatorname{im} d_q \overset{d_q}{\leftarrow} C_q \overset{d_{q+1}}{\leftarrow} C_{q+1}/\ker d_{q+1} \leftarrow 0$$

but because of the exactness of the long sequence, we can write as well:

$$0 \leftarrow \ker d_{q-1} \overset{d_q}{\leftarrow} C_q \overset{d_{q+1}}{\leftarrow} \operatorname{coker} d_{q+2} \leftarrow 0$$

This "justifies" the standard use of the long exact sequences: if a long exact sequence C_* is given and if for every q the chain groups C_{3q+1} and C_{3q+2} are known:

$$\cdots \leftarrow \underset{\text{known}}{C_{3q-2}} \overset{d_{3q-1}}{\leftarrow} \underset{\text{known}}{C_{3q-1}} \overset{d_{3q}}{\leftarrow} \underset{???}{C_{3q}} \overset{d_{3q+1}}{\leftarrow} \underset{\text{known}}{C_{3q+1}} \overset{d_{3q+2}}{\leftarrow} \underset{\text{known}}{C_{3q+2}} \leftarrow \cdots$$

and also the morphisms d_{3q+2}, then the chain group C_{3q} is an extension of $\ker d_{3q-1}$ by $\operatorname{coker} d_{3q+2}$:

$$0 \leftarrow \ker d_{3q-1} \leftarrow C_{3q} \leftarrow \operatorname{coker} d_{3q+2} \leftarrow 0$$

You understand you need to *know* the maps d_{3q+2} for every q to determine such kernels and cokernels, and when this is done, there remains an extension problem.

In simple situations, this is easy. For example if every d_{3q+2} is known to be an isomorphism, then kernels and cokernels are null and $C_{3q} = 0$. Another case is when every C_{3q-1} (resp. C_{3q+1}) is null, then $C_{3q} \cong C_{3q+1}$ (resp. C_{3q-1}).

But in general, the problem is highly non-trivial. Difference between *effective* homology and ordinary homology consists in particular in being permanently *vigilant* to be able to determine the maps d_{3q+2} and to have sufficient data to solve the extension problem.

2.6 The long exact sequence of a short exact sequence.

It is a short exact sequence *of chain-complexes* which produces a long exact sequence.

Theorem 22 [36, II.4.1] — *Let:*

$$0 \leftarrow A_* \overset{j}{\leftarrow} B_* \overset{i}{\leftarrow} C_* \leftarrow 0$$

a short exact sequence of chain-complexes. *Then a canonical long exact sequence of modules is obtained:*

$$\cdots \leftarrow H_{q-1}(C_*) \overset{\partial}{\leftarrow} H_q(A_*) \overset{j}{\leftarrow} H_q(B_*) \overset{i}{\leftarrow} H_q(C_*) \overset{\partial}{\leftarrow} H_{q+1}(A_*) \leftarrow \cdots$$

A short exact sequence of chain-complexes is a large diagram:

$$\begin{tikzcd}
 & {} \arrow[d, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & \\
0 & A_{q+1} \arrow[l] \arrow[d] & B_{q+1} \arrow[l, "j"'] \arrow[d] & C_{q+1} \arrow[l, "i"'] \arrow[d] & 0 \arrow[l] \\
0 & A_q \arrow[l] \arrow[d] & B_q \arrow[l, "j"'] \arrow[d] & C_q \arrow[l, "i"'] \arrow[d] & 0 \arrow[l] \\
0 & A_{q-1} \arrow[l] \arrow[d, dotted] & B_{q-1} \arrow[l, "j"'] \arrow[d, dotted] & C_{q-1} \arrow[l, "i"'] \arrow[d, dotted] & 0 \arrow[l] \\
 & {} & {} & {} &
\end{tikzcd}$$

where all the horizontal short sequences are exact, and the three vertical sequences are chain-complexes. It is understood i and j are *chain-complex morphisms*, that is, every square of our diagram is commutative.

PROOF. See [36, II.4.1]. It is a matter of *diagram chasing* in our diagram. The *connection morphism* for example $\partial : H_{q+1}(A_*) \to H_q(C_*)$ is of particular interest. It comes from a diagram of objects:

$$\begin{tikzcd}
\mathfrak{h}_{q+1} \ni z_{q+1} \arrow[d, mapsto] & c_{q+1} \arrow[l, mapsto] \arrow[d, mapsto] & \\
0 & b_q \arrow[l, mapsto] \arrow[d, mapsto] & z_q \in \mathfrak{h}_q \arrow[l, mapsto] \arrow[d, mapsto] \\
 & 0 & 0 \arrow[l, mapsto]
\end{tikzcd}$$

obtained as follows. Let $\mathfrak{h}_{q+1} \in H_{q+1}(A_*)$ be a homology class of A_* of degree $q+1$. Let $z_{q+1} \in A_{q+1}$ be a cycle representing $\mathfrak{h}_{q+1}$: the image in A_q by the vertical boundary map is null. Because every j is surjective, we can find a chain $c_{q+1} \in B_{q+1}$ which is a j-preimage of z_{q+1}. Then the vertical image b_q of c_{q+1} must satisfy $j(b_q) = 0$, for the left hand square is commutative. Exactness of the horizontal row implies there is a unique i-preimage $z_q \in C_q$. The right hand square is also commutative. The vertical image of b_q is null ($dd = 0$), so that, taking account of the injectivity of i, the vertical image of z_q is also null: z_q is a cycle which defines a homology class $\mathfrak{h}_q \in H_q(C_*)$. If c'_{q+1} is another choice instead of c_{q+1} for a preimage of z_{q+1}, then this generates in the same way b'_q, z'_q and $\mathfrak{h}'_q$ but in fact $\mathfrak{h}_q = \mathfrak{h}'_q$, which results from the other diagram and analogous arguments:

$$\begin{tikzcd}
0 \arrow[d, mapsto] & c'_{q+1} - c_{q+1} \arrow[l, mapsto] \arrow[d, mapsto] & c''_{q+1} \arrow[l, mapsto] \arrow[d, mapsto] \\
0 & b'_q - b_q \arrow[l, mapsto] & z'_q - z_q \arrow[l, mapsto]
\end{tikzcd}$$

You must also prove the independance with respect to the choice of $z_{q+1} \in H_{q+1}(A_*)$, analogous exercise. The connexion map $\partial : \mathfrak{h}_{q+1} \mapsto \mathfrak{h}_q$ so defined is a module morphism — exercise — and you must construct the other (induced) morphisms i and j of the long exact sequence — exercises. You must prove this long sequence actually is… exact. For example let us examine the exactness in $H_{q+1}(A_*)$. If ever $\mathfrak{h}_{q+1}$ is the image of $\mathfrak{h}'_{q+1} \in H_{q+1}(B_*)$, we may choose $c_{q+1} = z'_{q+1} \in \mathfrak{h}'_{q+1}$, it is a cycle and $b_q = 0$:

$$\begin{tikzcd} z_{q+1} & z'_{q+1} \arrow[l, mapsto] \arrow[d, mapsto] & \\ & b_q = 0 & z_q = 0 \arrow[l, mapsto] \end{tikzcd}$$

so that $\mathfrak{h}_q = 0$. Conversely, if $\mathfrak{h}_q = 0$, this means the final cycle z_q is a boundary:

$$\begin{tikzcd} z_{q+1} & c_{q+1} \arrow[l, mapsto] \arrow[d, mapsto] & c'_{q+1} \arrow[d, mapsto] \\ & b_q & z_q \arrow[l, mapsto] \end{tikzcd}$$

But this implies you have also this diagram:

$$\begin{tikzcd} 0 & c''_{q+1} \arrow[l, mapsto] \arrow[d, mapsto] & c'_{q+1} \arrow[l, mapsto] \arrow[d, mapsto] \\ & b_q & z_q \arrow[l, mapsto] \end{tikzcd}$$

Now $c_{q+1} - c''_{q+1}$ is another choice for c_{q+1}, a choice which is a (vertical) cycle, therefore defining a homology class $\mathfrak{h}'_{q+1}$ satisfying $j(\mathfrak{h}'_{q+1}) = \mathfrak{h}_{q+1}$. If it is the first time you practice this sport, you must carefully examine all the details of the other components of the proof. ■

We will see later, cf. Definition 80, that in *effective homology*, the analogous theorem needs a further hypothesis: the exactness property of the short exact sequence of chain-complexes must be *effective*: an *algorithm* must be present in the environment *returning* (producing) the various preimages which are required; it happens it is always the case in the practical applications. And the demonstration is then much easier and, very important, other *algorithms* are produced making *effective* the exactness property of the resulting long exact sequence.

2.6.1 Examples.

Definition 23 — A *simplicial pair* (K, L) *is a pair made of a simplicial complex* K *and a simplicial subcomplex* L *of* K.

The vertex set V_L of L is a subset $V_L \subset V_K$ of the vertex set of K, the same for the simplices. For example let us define the (simplicial) $(n-1)$-sphere as the simplicial complex $S^{n-1} = (\underline{n}, \mathcal{P}_*(\underline{n}) - \{\underline{n}\})$. A simplex is an arbitrary subset of $\underline{n}$

except the void subset $\emptyset$ and the full subset $\underline{n}$. For example the 2-sphere is:

$$S^2 = (\underline{3}, \left\{ \begin{array}{l} (0),(1),(2),(3), \\ (0,1),(0,2),(0,3),(1,2),(1,3),(2,3), \\ (0,1,2),(0,1,3),(0,2,3),(1,2,3) \end{array} \right\})$$

It is the *hollow* tetrahedron, while Δ^3 is the *solid* tetrahedron. In general S^{n-1} is a simplicial subcomplex of the standard n-simplex Δ^n, and (Δ^n, S^{n-1}) is a simplcial pair.

Definition 24 — Let (K, L) be a *simplicial pair*. The *relative chain-complex* $C_*(K, L)$ is the quotient complex $C_*(K, L) = C_*(K)/C_*(L)$. The *relative homology* $H_*(K, L)$ accordingly is $H_*(K, L) := H_*(C_*(K)/C_*(L))$.

The second component L is a simplicial subcomplex of the first one K, so that the corresponding chain-complex $C_*(L)$ is a sub-chain-complex of $C_*(K)$, both differentials are compatible, which allows us to define the quotient chain-complex $C_*(K)/C_*(L)$ and the relative homology is the homology of this quotient.

Theorem 25 — *If (K, L) is a simplicial pair, then a long exact sequence is obtained:*

$$\cdots \leftarrow H_{q-1}(L) \stackrel{\partial}{\leftarrow} H_q(K, L) \stackrel{j}{\leftarrow} H_q(K) \stackrel{i}{\leftarrow} H_q(L) \stackrel{\partial}{\leftarrow} H_{q+1}(K, L) \leftarrow \cdots$$

Note in particular the tempting result $H_q(K, L) \cong H_q(K)/H_q(L)$ not only in general is false, but it does not make sense: in general no inclusion relation between $H_q(L)$ and $H_q(K)$. The inclusion relations $C_q(L) \subset C_q(K)$, $Z_q(L) \subset Z_q(K)$ and $B_q(L) \subset B_q(K)$ are true, a canonical map $H_q(L) \rightarrow H_q(L)$ therefore is defined, but this map is not in general injective[8].

Proof. For every q, we have a short exact sequence:

$$0 \leftarrow C_q(K)/C_q(L) \stackrel{j}{\leftarrow} C_q(K) \stackrel{i}{\leftarrow} C_q(L) \leftarrow 0$$

But the maps i and j are compatible with the differentials of the chain-complexes, so that we have in fact a short exact sequence *of chain-complexes*:

$$0 \leftarrow C_*(K)/C_*(L) \stackrel{j}{\leftarrow} C_*(K) \stackrel{i}{\leftarrow} C_*(L) \leftarrow 0$$

and there remains to apply Theorem 22. ■

Proposition 26 — *Let S^{n-1} be the $(n-1)$-sphere and $\mathfrak{R}$ be the ground ring. Then, if $n \geq 2$, the homology groups $H_q(S^{n-1})$ are null except $H_0(S^{n-1}) = H_{n-1}(S^{n-1}) = \mathfrak{R}$.*

[8] $3 \leq 6$ and $2 \leq 6$ do not imply $3/2 \leq 6/6$.

PROOF. Let us consider the pair (Δ^n, S^{n-1}). Then all the chain groups of $C_*(\Delta^n)/C_*(S^{n-1})$ are null except $C_n(\Delta^n)/C_n(S^{n-1}) = \mathfrak{R}$: only the maximal simplex of Δ^n is not in S^{n-1}. So that all the relative homology groups $H_q(\Delta^n, S^{n-1})$ are null except $H_n(\Delta^n, S^{n-1}) = \mathfrak{R}$. Now in the long exact sequence connecting $H_*(\Delta^n)$ (known), $H_*(\Delta^n, S^{n-1})$ (known) and $H_*(S^{n-1})$ (unknown), there are essentially two interesting sections:

$$[H_0(\Delta^n, S^{n-1}) = 0] \leftarrow [H_0(\Delta^n) = \mathfrak{R}] \leftarrow [H_0(S^{n-1}) = ?] \leftarrow [H_1(\Delta^n, S^{n-1}) = 0]$$
$$[H_{n-1}(\Delta^{n-1}) = 0] \leftarrow [H_{n-1}(S^{n-1}) = ?] \xleftarrow{\partial} [H_n(\Delta^n, S^{n-1}) = \mathfrak{R}] \leftarrow [H_n(\Delta^n) = 0]$$

The extreme modules are null and, because of exactness, the central morphisms are isomorphisms[9]. ■

Note also the connexion morphism ∂ allows us to identify a canonical representant (in fact unique up to sign, why?) for a generator of $H_{n-1}(S^{n-1})$, namely the boundary of the maximal simplex $(0, \ldots, n)$ of Δ^n; note this maximal simplex does not live in S^{n-1}, but its boundary yes.

2.7 About computability.

All these didactical examples involve *finite* simplicial complexes, so that no theoretical computability problem here. However the benefit of the various explained methods is already clear. For example let us take the standard simplex Δ^{10}. If you want to compute $H_5(\Delta^{10}) = 0$ by brute force, the boundary matrices to be considered are 462×462 and 462×330 so that proving kernel = image by "simple" computation is already a little serious. Moreover, when we will ask for *constructive* homology, see Section 4.3, we will have to be ready to quickly return a boundary preimage for every cycle, for this cycle is certainly homologous to 0. But Theorem 18 gives immediately the answer: this theorem in fact gives a *reduction* (see Definition 42) $C_*(\Delta_n) \Rrightarrow C_*(\Delta^0)$, so that the homological problem for Δ^n is equivalent to the same problem for Δ^0, which problem is very simple.

This is a common situation. Even when the theoretical computability problem has a trivial solution, an appropriate theoretical study of this computability problem can produce dramatically better solutions. Another typical example is the computation of the homology groups of the Eilenberg-MacLane spaces $K(\pi, n)$ for π an Abelian group of finite type. The general results quickly sketched after Theorem 150 prove the effective homology of these spaces is computable. In the particular case where π is a *finite* Abelian group, the brute result is trivial, but the general method deduced from Theorem 150 remains essential for concrete computations. Let us consider for example the group $H_8(K(\mathbb{Z}_2, 4)) = \mathbb{Z}_4$. A "direct" computation would require $n_7 \times n_8$ and $n_8 \times n_9$ matrices with:

$$\begin{aligned} n_7 &= 34359509614 \\ n_8 &= 1180591620442534312297 \\ n_9 &= 85070591730234605240519066638188154620 \end{aligned}$$

[9] What about the case $n = 1$?

The method resulting from Theorem 150 reduces the problem to a smaller chain-complex with the analogous dimensions being $n'_7 = 4$, $n'_8 = 8$ and $n'_9 = 15$. The result is then obtained in less than 2 seconds with a modest laptop, most computing time being devoted to *compute* these small matrices, which remains a non-trivial task.

But the most striking results which are obtained in *constructive* homological algebra concern cases where the studied chain-complex defining homology-groups is *not* of finite type. It is the general situation for loop spaces leading to our simple solution for Adams' problem, see Section 9.5. For Eilenberg-MacLane spaces, if you are interested by $H_8(K(\mathbb{Z}, 4)) = \mathbb{Z}_3 + \mathbb{Z}$, then the corresponding numbers n_7, n_8 and n_9 are *infinite*. Eilenberg and MacLane in their wonderful papers [20, 21] explained how to obtain an equivalent chain-complex of finite type (in this case $n'_7 = 1$ and $n'_8 = n'_9 = 2$) giving the right homology groups; it was the first historical case where *constructive* homological algebra was implicitly used, without any constructive terminology... The matter of these notes consists in a systematic extension of these constructive methods, producing results with a very general scope. The strong difference with the general style of Eilenberg-MacLane's work is that we will have to *keep in our environment* the original $K(\mathbb{Z}, 4)$ itself, with a functional implementation, as a *locally effective* object, for example to be able to compute a spectral sequence where this object is involved.

3 Spectral sequences.

3.1 Introduction.

The previous section explained how the long exact sequence of a short exact sequence of chain-complexes can be used to determine some unknown homology groups. The typical case being the last example: three chain-complexes are present in the environment: $C_*(\Delta^n)$, $C_*(\Delta^n, S^{n-1})$ and $C_*(S^{n-1})$. We knew the homology groups $H_*(\Delta_n)$ and $H_*(\Delta^n, S^{n-1})$ and the long exact sequence allowed us to obtain the unknown groups $H_*(S^{n-1})$.

This is the general process in the computation of homology groups, and the same for homotopy groups in Algebraic Topology. Objects more and more complicated are considered, and the invariants of the new objects are obtained from invariants of simpler previous ones and of a careful study of the "difference".

But this description unfortunately is simplistic. For example if you know $H_*(K)$ and $H_*(K, L)$, and you try to deduce $H_*(L)$, the long exact sequence:

$$\cdots \leftarrow H_q(K, L) \overset{j}{\leftarrow} H_q(K) \overset{i}{\leftarrow} H_q(L) \overset{\partial}{\leftarrow} H_{q+1}(K, L) \overset{j}{\leftarrow} H_{q+1}(K) \leftarrow \cdots$$

produces a short exact sequence:

$$0 \leftarrow \ker j \overset{i}{\leftarrow} H_q(L) \overset{\partial}{\leftarrow} \operatorname{coker} j \leftarrow 0$$

and if you *do not know* the exact nature of the map j, you cannot proceed; as soon as the situation becomes a little more complicated, it is the *most frequent* case. And even if you can determine the groups $\ker j$ and $\operatorname{coker} j$, there remains a possible extension problem needing also other informations to be solved.

Claim 27 — *Except in… exceptional situations, the long exact sequence of homology* <u>*is not*</u> *an algorithm allowing to compute an unknown group when the four neighbouring groups are known.*

And most books about homological agebra do not give any explanations about this lack in the theory; they even give frequently the unpleasant feeling that they *hide* this deficiency, but more probably the authors do not have a sufficiently precise knowledge of the very nature of the constructive requirement.

The present text is exactly devoted to provide the missing tools allowing to transform usual homological algebra into a modern *constructive* theory. Experience shows it is quite elementary, but two essential notions are required. From an algorithmic point of view, *high level functional programming* is definitively necessary; fortunately, standard computer science knows this matter from a long time, and the modern application tools are the so called functional programming languages such as Lisp, ML, Haskell, with powerful compilers. From an "ordinary" mathematical point of view, the *basic perturbation lemma* (Henri Cartan, Shih Weishu [61], Ronnie Brown [11]) is the key point.

Which probably explains the terrible delay of homological algebra with respect to the modern constructive point of view is the fact that the elementary results explained here to satisfy constructiveness cannot be seriously used *without machines and programs.* The analogy with commutative algebra thirty years ago is striking. Noone can now hope to work a long time in commutative algebra without using the Groebner bases. Groebner bases are elementary, but cannot be used without auxiliary machines and programs. Groebner bases are quite elementary, the same for the homological perturbation lemma. More precisely the basic theory of Groebner bases is elementary, but looking for more and more efficient implementations, in particular for special cases, remains an active research subject. And the situation is the same for the homological perturbation lemma.

This section is devoted to a short presentation of the *spectral sequence* theory, and the situation for spectral sequences is the same as for the long exact sequence. In *exceptional* cases, a spectral sequence *can be* a process giving unknown homology groups when other homology groups are given, but in the general situation, the constructive requirement is not satisfied: no *general algorithm* can be deduced from the spectral sequence theory. The homological perturbation lemma will allow us to replace the usual spectral sequences by *effective* versions. Which is quite amazing in this case is the fact that these effective versions are *terribly simpler* to design than the ordinary spectral sequences, but, think of the Groebner bases, these effective spectral sequences cannot be used without the corresponding machines and programs.

The spectral sequences are also used in commutative algebra, because of the frequent presence of Koszul complexes playing an important role through their homology groups. We will see the point of view presented here gives also very interesting results in commutative algebra, mainly to compute the *effective* homology of Koszul complexes, richer than the ordinary homology; for example this effective homology gives a direct method to compute the minimal resolution of modules with respect to polynomial rings, known as an essential and difficult problem.

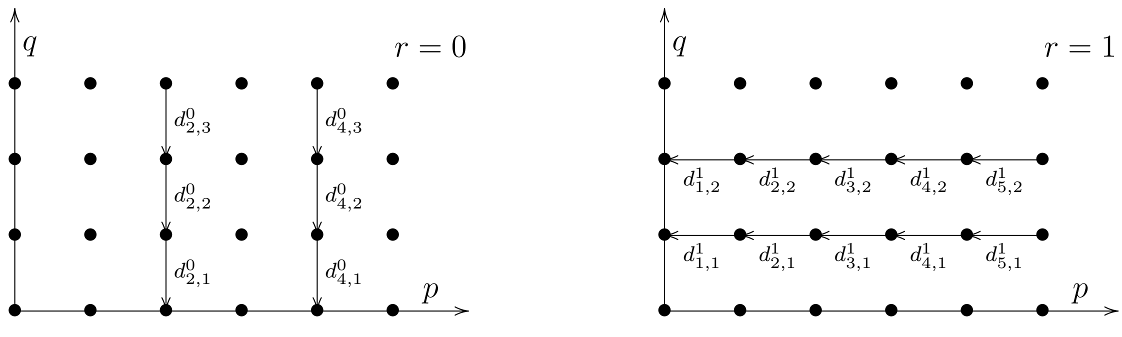
3.2 Notion of spectral sequence.

One of the best references to attack the subject is [36, Chapter XI]. The *didactic* quality of this text is the highest we know. In particular MacLane begins to explain how to *use* a spectral sequence before proving its construction, a wise organization. We just give here a short presentation of the general structure of spectral sequences, advising the reader to study [36, Chapter XI] for further details and results. The most complete reference about spectral sequences of course is [42].

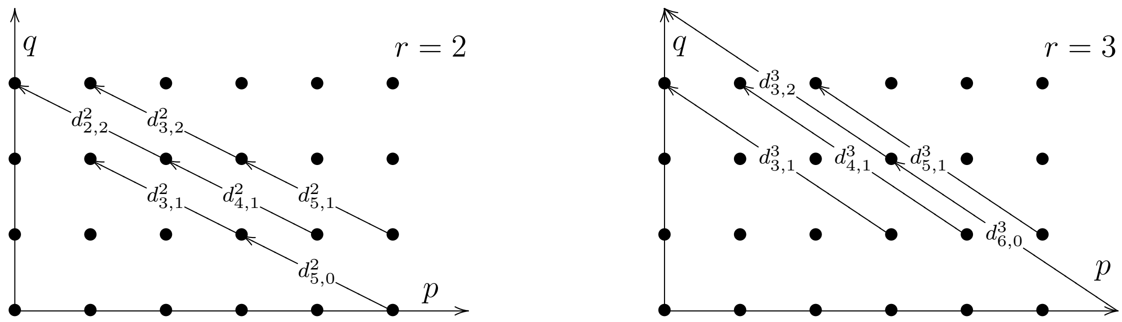
Definition 28 — A *spectral sequence* is a collection $\{E^r_{p,q}, d^r_{p,q}\}^{r\geq r_0}_{p,q\in\mathbb{Z}}$ satisfying the following properties:

- Every $E^r_{p,q}$ is an $\mathfrak{R}$-module ($\mathfrak{R}$ is the underlying ground ring).
- Every $d^r_{p,q}$ is a morphism $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$.
- Every composition $d^r_{p,q} d^r_{p+r,q-r+1}$ is null, so that a homology group $H^r_{p,q} = \ker d^r_{p,q} / \operatorname{im} d^r_{p+r,q-r+1}$ is defined.
- For every $r \geq r_0$, $p, q \in \mathbb{Z}$, an isomorphism $H^r_{p,q} \cong E^{r+1}_{p,q}$ is provided.

A geometric representation of the notion of spectral sequence is very useful. Look at this figure[10]:



[10]Strongly inspired by the analogous scheme of [36, Section XI.1], without any kind permission of Springer-Verlag.



You must consider the integer parameter r as a discrete time, a spectral sequence can be thought of as a dynamical system. The figures represent the state of our system at times 0, 1, 2 and 3. Usually the initial time r_0 is 0, 1 or 2 and we will mot mention it anymore. A convenient terminology consists in considering a spectral sequence as a book where the page r is visible at time r. The page r is made of a collection of modules $\{E^r_{p,q}\}_{p,q\in\mathbb{Z}}$; every morphism $d^r_{p,q}$ starts from $E^r_{p,q}$ and goes to $E^r_{p-r,q+r-1}$: the shift for the *horizontal* degree p is $-r$, the page number, and the shift for the *total* degree $p+q$ always is -1, so that the shift for the *vertical* degree q necessarily is $r-1$. On the above figures, only a few differentials are displayed.

Because of the rule about the composition of two successive $d^r_{p,q}$'s, every page is a collection of chain-complexes, where in the above representation the (oriented) "slope" is $(q-1)/(-p)$. Therefore the page r produces a collection of homology groups $H^r_{p,q}$ and $H^r_{p,q}$ is isomorphic to $E^{r+1}_{p,q}$, one usually says $H^r_{p,q}$ "is" $E^{r+1}_{p,q}$. In short, every page is a collection of chain-complexes and the collection of corresponding homology groups "is" the next page, but it is exactly at this point the constructiveness property in most situations fails, point examined later.

Very frequently the spectral sequence is null outside some quadrant of the (p,q)-plane; for example, if p or $q<0 \Rightarrow E^r_{p,q}=0$, one says it is a *first quadrant* spectral sequence; a *second quadrant* spectral sequence is null for $p>0$ or $q<0$.

Definition 29 — A spectral sequence $\{E^r_{p,q}, d^r_{p,q}\}$ is *convergent* if for every $p,q\in\mathbb{Z}$ the relations $d^r_{p,q}=0=d^r_{p+r,q-r+1}$ holds for $r\geq r_{p,q}$.

If the convergence property is satisfied, then $E^r_{p,q}=H^r_{p,q}$ "$=$" $E^{r+1}_{p,q}=\cdots$ for $r=r_{p,q}$.

Definition 30 — If a spectral sequence $\{E^r_{p,q}, d^r_{p,q}\}$ is convergent, $E^\infty_{p,q}:=$ "$\lim$"$_{r\to\infty}E^r_{p,q}$.

As usual, only the isomorphism class of the limit is defined. For example a first quadrant spectral sequence is necessarily convergent, because $r>p\Rightarrow d^r_{p,q}=0$ and $r>q+1\Rightarrow d^r_{p+r,q-r+1}=0$. A second quadrant spectral sequence is not necessarily convergent.

Definition 31 — Let $\{H_n\}_{n\in\mathbb{Z}}$ be a collection of modules, probably a collection of interesting homology groups. The spectral sequence $\{E^r_{p,q}, d^r_{p,q}\}$ *converges towards* $\{H_n\}_{n\in\mathbb{Z}}$ if the spectral sequence is convergent and if there exists a filtration $\{H_{p,q}\}_{p+q=n}$ of every H_n such that $E^\infty_{p,q} \cong H_{p,q}/H_{p-1,q+1}$. The collection $\{H_n\}_{n\in\mathbb{Z}}$ is then called the *abutment* of the spectral sequence.

The filtration of H_n must be coherent, that is $H_{p,q} \subset H_{p+1,q-1}$, $\cap_{p+q=n} H_{p,q} = 0$ and $\cup_{p+q=n} H_{p,q} = H_n$. It is an increasing filtration indexed on p, but it is convenient to recall the second index q, which also implicitly implies the total degree $n = p + q$. For example for a first quadrant spectral sequence, the context would imply $0 = H_{-1,n+1} \subset H_{0,n} \subset H_{1,n-1} \subset \cdots \subset H_{n,0} = H_n$.

There is a strange but convenient notation for such a convergence property:

$$E^r_{p,q} \Rightarrow H_{p+q}$$

The convergence is implicitly concerned by which happens when $r \to \infty$. The double arrow '$\Rightarrow$' instead of the simple one '$\to$' recalls the convergence property is quite complex. The ambiguous index of H_{p+q} means some filtration of H_n is involved correlated to the double indexation of $E^\infty_{p,q}$.

3.3 The Serre spectral sequence.

The Serre spectral sequence was invented in 1950 of course by Jean-Pierre Serre, using anterior works of Jean Leray and Jean-Louis Koszul; this spectral sequence allowed him to determine many homotopy groups, in particular sphere homotopy groups. This spectral sequence concerns the *fibrations*:

$$F \hookrightarrow E \to B$$

where F is the *fibre* space, B the *base* space and E the *total* space. These were initially topological spaces, but this notion of fibration can be generalized to many other situations. The total space E is to be considered as a *twisted product* of the base space B by the fibre space F. The underlying twisting operator τ is defined by different means according to the context, but the idea is constant: τ explains how the twisted product $E = F \times_\tau B$ is different from the trivial product $F \times B$, which product depends in turn on the category we are working in. See for example [64, Section I.2] for the original case of the *fibre bundles*; the twist then is a collection of *coordinate functions*.

Theorem 32 — *Let $E = F \times_\tau B$ be a topological fibration with a base space B simply connected. Then a first quadrant spectral sequence $\{E^r_{p,q}, d^r_{p,q}\}_{r\geq 2}$ is defined with $E^2_{p,q} = H_p(B; H_q(F))$ and $E^r_{p,q} \Rightarrow H_{p+q}(E)$.*

We are working in general topology and there is a process called *singular homology* associating to every topological space X, every integer n and every abelian group $\mathfrak{R}$ (here not necessarily a ring) a homology group $H_n(X;\mathfrak{R})$, the n-th homology group of X with coefficients in $\mathfrak{R}$. The process is strongly inspired by which had been done in Section 2.2.3, but adapted to an arbitrary topological space thanks to the notion of *singular simplex*, see for example [22, Chapter VII]. We will not be concerned by the (interesting) definition of the singular homology groups. It happens if the topological space X comes from a simplicial complex, the simplicial homology groups and the singular homology groups are canonically isomorphic.

The Serre spectral sequence establishes a rich set of relations between the homology groups $H_*(F)$, $H_*(E)$ and $H_*(B)$ of the fibre space, total space and base space of a fibration, at least when the base space is simply connected. It is frequently somewhat implicitly "suggested" this spectral sequence is a process allowing one for example to *compute* the groups $H_*(E)$ when the groups $H_*(B)$ and $H_*(F)$ are known. But in general this is false. In general the differentials $d^2_{p,q}$ are unknown, and even if you know them, you will be able to compute the $E^3_{p,q}$'s, but to continue the process, you need now the differentials $d^3_{p,q}$ and in general you do not have the necessary information to compute them. And so on.

And if by any chance you reach the limit groups $E^\infty_{p,q}$, you have the group $H_{0,n} = E^\infty_{0,n}$, but to determine the next component of the filtration of H_n, the exact sequence:

$$0 \leftarrow E^\infty_{1,n-1} \leftarrow H_{1,n-1} \leftarrow H_{0,n} \leftarrow 0$$

shows $H_{1,n-1}$ is the solution of an extension problem which can be very difficult, we will show a typical example. And if you succeed, again an extension problem for $H_{2,n-2}$, and so on...

Claim 33 — *Let $F \hookrightarrow E \rightarrow B$ be a given fibration with B simply connected. Except in... exceptional situations, the Serre spectral sequence <u>is not</u> an algorithm allowing to compute $H_*(E)$ when $H_*(B)$ and $H_*(F)$ are known. More generally, except in exceptional situations, the page $r+1$ of a spectral sequence cannot be deduced from the page r and the other available data.*

These negative appreciations of course must not reduce the interest of the various known spectral sequences. The point of view used here is the following: yes the spectral sequences are in many circumstances quite essential, yes they allowed to obtain many very interesting results, but their general organisation is not algorithmic; how this deficiency with respect to usual modern mathematics could be corrected? In short, how a spectral sequence can be made *constructive*? It is our main concern.

3.3.1 A positive example.

When writing these notes, MacLane's excellent book [36] is not far and instead of considering the loop spaces of spheres, the first example of this book, we use the symmetrical example of $B\mathbb{H}_* = P^\infty\mathbb{H}$, the classifying space of the multiplicative group $\mathbb{H}_*$ of the quaternion field $\mathbb{H}$, in other words the infinite quaternionic projective space. The topological group $\mathbb{H}_*$ automatically generates [43] a universal principal fibration:

$$\mathbb{H}_* \hookrightarrow E\mathbb{H}_* \to B\mathbb{H}_*.$$

This means our group $\mathbb{H}_*$ freely acts on the total space $E\mathbb{H}_*$, the base space $B\mathbb{H}_*$ being the corresponding homogeneous space $B\mathbb{H}_* = E\mathbb{H}_*/\mathbb{H}_*$. Saying the fibration is *universal* amounts to requiring the total space $E\mathbb{H}_*$ is *contractible*, that is, has the homotopy type of a point, which needs a few definitions to be understood.

Definition 34 — Two continuous maps $f_0, f_1 : X \to Y$ are *homotopic* if there exists a continuous map $F : X \times [0,1] \to Y$ such that $f_0(x) = F(x,0)$ and $f_1(x) = F(x,1)$ for every $x \in X$.

In other words, two continuous maps f_0 and f_1 are homotopic if a continuous *deformation* F can be installed between them.

Theorem 35 [22, Section VII.7] — *If two continuous maps $f, g : X \to Y$ are homotopic, then the induced maps $f_*, g_* : H_*(X;\mathfrak{R}) \to H_*(Y;\mathfrak{R})$ between singular homology groups, with respect to an arbitrary coefficient group $\mathfrak{R}$, are equal.*

Definition 36 — A continuous map $f : X \to Y$ is a *homotopy equivalence* if there exists another continuous map $g : Y \to X$ such that gf is homotopic to id_X and fg is homotopic to id_Y.

Definition 37 — Two topological spaces X and Y have the *same homotopy type* if there exists a homotopy equivalence $f : X \to Y$.

A homotopy equivalence $f : X \to Y$ therefore induces isomorphisms $f : H_*(X) \xrightarrow{\cong} H_*(Y)$. The same homotopy type requires isomorphic homology groups, but unfortunately the converse is false: it is an *open problem* to give *computable* characteristic conditions for homotopy equivalence. It is generally "understood" such a condition is given by the so-called *Postnikov-invariants* or *k-invariants*, but this is false [54].

Definition 38 — A topological space X is *contractible* if it has the homotopy type of a point.

Definition 39 — If G is a topological group, a principal fibration:

$$G \hookrightarrow EG \to BG$$

is *universal* if the total space EG is contractible [64, 30]. It is then proved the homotopy type of the so-called *classifying space* BG is well defined up to homotopy.

A point $*$[11] is a (multiplicative) *unit* in the topological world: the product $* \times X$ is canonically homeomorphic to X. The total space EG of a universal fibration is some twisted product $EG = G \times_\tau BG$, and because this product has the homotopy type of a point, the classifying space BG can be understood as a "twisted inverse" of the initial group G, but *up to homotopy*. Such a twisted inverse is itself unique up to homotopy.

These classifying spaces BG are very important and the computation of their homology groups as well. The dual notion of *cohomology groups* of these classifying spaces leads to the important notion of characteristic classes of principal fibrations [44]. And it is essential to be able to compute the homology groups of classifying spaces.

In the case of our quaternionic multiplicative group $\mathbb{H}_*$, a radial homotopy easily allows to prove the inclusion $S^3 \hookrightarrow \mathbb{H}_*$ is a homotopy equivalence, so that the homology groups of $\mathbb{H}_*$ and S^3 are the same. Proposition 26 proves the *simplicial* homology groups $H_n(S^3;\mathbb{Z})$ are null except $H_0(S^3) = H_3(S^3) = \mathbb{Z}$. And the isomorphism theorem between singular and simplicial homology groups [22, Section VII.10] implies it is the same for the singular homology groups, so that $H_n(\mathbb{H}_*) = 0$ except $H_0(\mathbb{H}_*) = H_3(\mathbb{H}_*) = \mathbb{Z}$.

It is convenient to shorten $\mathbb{H}_* =: G$, $E\mathbb{H}_* =: E$ and $B\mathbb{H}_* =: B$, so that the diagram:

$$G \hookrightarrow E \rightarrow B$$

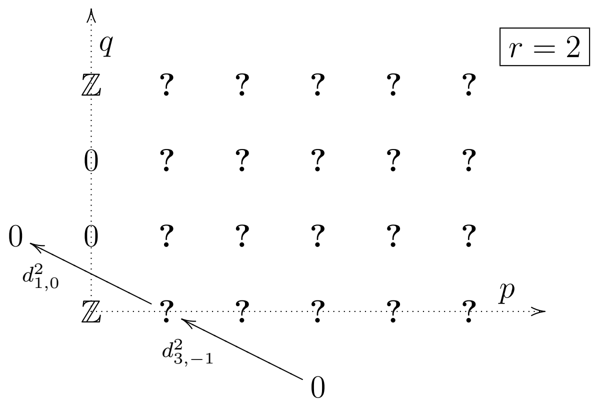
denotes now our specific universal fibration around the topological group $G = \mathbb{H}_*$.

Because the total space E is contractible, all its homology groups are null except $H_0(EG) = \mathbb{Z}$. Knowing the groups $H_*(G)$ and $H_*(E)$, the game now consists in guessing the groups $H_*(BG)$.

The Serre spectral sequence of a fibration involving G, E and B describes $E^2_{p,q} = H_p(B; H_q(G))$; in general the *universal coefficient theorem* [36, Section V.11] allows to deduce the groups $H_n(X;\mathfrak{R})$, where $\mathfrak{R}$ is an arbitrary abelian group, from the *integer* homology groups $H_n(X;\mathbb{Z})$ most often denoted by $H_n(X)$ in short. Here the situation is simple: $H_p(BG; H_q(G)) = 0$ except for $q = 0$ or 3 where $H_p(BG; H_q(G)) = H_p(BG;\mathbb{Z})$. In particular $H_0(BG, H_q(G)) = 0$ except for $q = 0$ or 3 where the value is $\mathbb{Z}$; this is because BG is necessarily connected, which implies $H_0(BG;\mathbb{Z}) = \mathbb{Z}$.

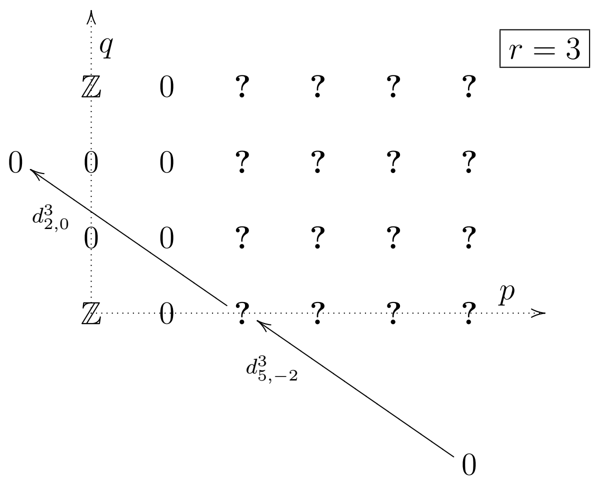
The initial state of our study is the known state of the page 2 of our spectral sequence:

[11] Not to be confused with a generic index such as in $H_*(X)$.



It is a first quadrant spectral sequence, so that the d^2-arrows arriving and starting from $E^2_{1,0}$ necessarily are null. This entails $E^3_{1,0} = \ker d^2_{1,0} / \operatorname{im} d^2_{3,-1} = E^2_{1,0}/0 = E^2_{1,0}$. The same for the next r's, and $E^2_{1,0} = E^3_{1,0} = \cdots = E^\infty_{1,0}$. At the abutment of the spectral sequence, we know all the $H_n(EG)$ are null for $n > 0$, so that certainly all the corresponding $E^\infty_{p,q} = H_{p,q}(EG)/H_{p-1,q+1}(EG)$ also are null. This implies that when $E^r_{p,q}$ becomes fixed, that is when $r > \max(p, q+1)$, the relation $E^r_{p,q} = 0$ is satisfied: for every (p, q) with p or $q > 0$, the spectral group $E^r_{p,q}$ must "die".

But for $E^2_{1,0}$, it must be already died at time $r = 2$, otherwise $E^2_{1,0} = E^\infty_{1,0} \neq 0$. We have proved $E^2_{1,0} = H_1(BG) = 0$. Now $E^2_{1,q} = H_1(BG; H_q(G)) = 0$, because of the universal coefficient theorem. So that we obtain this partial description for the page 3 of our spectral sequence.



This argument can be repeated for the column 2, starting this time from $E^2_{2,0}$, and also for the column 3, starting from $E^2_{3,0}$ and in this case, taking account of $E^2_{1,1} = 0$. We obtain $H_2(BG) = H_3(BG) = 0$. But for $E^r_{4,0}$, there is something new when $r = 4$:

$$\begin{array}{c|cccccc}
q & & & & & & \boxed{r=4} \\
\mathbb{Z} & 0 & 0 & 0 & ? & ? & \\
0 & 0 & 0 & 0 & ? & ? & \\
0 & 0 & 0 & 0 & ? & ? & \\
\mathbb{Z} & 0 & 0 & 0 & ? & ? & p \\
\end{array}
\qquad d^4_{4,0}$$

The group in position $(0, 3)$ starts non-null at time 2: $E^2_{0,3} = \mathbb{Z}$ and it must also die. Because the columns 2 or 3 are null, this group can be killed only at time 4, which implies the arrow $d^4_{4,0} : E^4_{4,0} \to E^4_{0,3}$ necessarily is an isomorphism. So that $E^4_{4,0} = \mathbb{Z}$ and going back to time 2, $H_4(BG) := H_4(BG, \mathbb{Z}) = H_4(BG, H_0(G)) = \mathbb{Z}$. The conclusion is the following: The column 4 for $r = 2, 3, 4$ is made of $E^r_{4,q} = \mathbb{Z}$ for $q = 0$ or 3, $E^r_{4,q} = 0$ for $q \neq 0$ and 3. But 4 is also the last time where it is possible to kill $E^4_{4,3} = \mathbb{Z}$, which implies by the same argument $H_8(BG) = \mathbb{Z}$. Finally we have proved:

$$\begin{array}{c|ccccccccc}
q & & & & & & & & & \boxed{r=4} \\
\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & p \\
\end{array}
\qquad d^4_{4,0}\cong \quad d^4_{8,0}\cong$$

Theorem 40 — *The homology groups $H_{4n+k}(B\mathbb{H}_*)$ are null for $k = 1, 2$ or 3 and the groups $H_{4n}(B\mathbb{H}_*)$ are all equal to $\mathbb{Z}$.*

Because of the very specific situation, this is a (rare) case where the spectral sequence can be entirely described: $E^r_{p,q} = \mathbb{Z}$ if $[p = q = 0$ and $2 \leq r \leq \infty]$ or $[p = 4n$ and $q = (0$ or $3)$ and $2 \leq r \leq 4]$. Otherwise every $E^r_{p,q} = 0$. The only non null $d^r_{p,q}$'s occur for $r = 4$, $p = 4n$ and $q = 0$ and they are isomorphisms $d^4_{4n,0} : E^4_{4n,0} = \mathbb{Z} \xrightarrow{\cong} E^4_{4n-4,3} = \mathbb{Z}$.

3.3.2 A negative example.

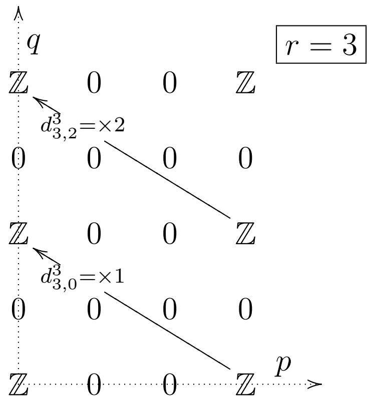
Jean-Pierre Serre got one of the 1954 Fields Medal, mainly for his computations of sphere homotopy groups, where the principal tool was his famous spectral sequence. To illustrate the non-constructive nature of this spectral sequence, we describe the beginning of his computations, up to the first point where the method failed.

If $(X, *)$ is a *based* topological space, that is, some *base point* $* \in X$ is given, the set of homotopy classes of continuous maps $\pi_n(X) := [(S^n, *), (X, *)]$ has a natural commutative group structure for $n \geq 2$ and it is a popular sport in algebraic topology to find out the groups $\pi_n(S^k)$. It is not hard to prove $\pi_n(S^k) = 0$ for $n < k$, $\pi_n(S^n) = \mathbb{Z}$, and the first event in the story was the amazing discovery by Hopf in 1935 that $\pi_3(S^2) = \mathbb{Z}$. In 1937, Freudenthal proved $\pi_4(S^2) = \mathbb{Z}_2$ (in algebraic topology, it is common to denote $\mathbb{Z}_2$ the quotient group $\mathbb{Z}/\mathbb{Z}_2$, not the p-adic ring!), and Serre at the beginning of the fifties computed many sphere homotopy groups; in particular he proved $\pi_6(S^3)$ has 12 elements, but could not choose between $\mathbb{Z}_{12}$ and $\mathbb{Z}_2 + \mathbb{Z}_6$. We want to describe the point where the spectral sequence method fails.

To compute $\pi_4(S^3)$, we can proceed as follows. We consider a fibration:

$$F_2 \hookrightarrow X_4 \to S^3$$

where $F_2 := K(\mathbb{Z}, 2)$ is an *Eilenberg-MacLane space*, in this case a space where every homotopy group is null except $\pi_2(F_2) = \mathbb{Z}$; such a space is well defined up to homotopy, it happens we can take $F_2 = P^\infty\mathbb{C}$. The beginning of the spectral sequence uses the homology of S^3, null except $H_0(S^3) = H_3(S^3) = \mathbb{Z}$ and the homology of F_2, null except $H_{2q}(F_2) = \mathbb{Z}$ for every $q \geq 0$. The critical page of the spectral sequence is the page $r = 3$:



Our fibration is not completely defined, we have not explained how the twisting operator τ of $X_4 = F_2 \times_\tau S^3$ is defined. We do not want to give the details, but the twisting operator τ is entirely defined[12] by the fact the arrow $d^3_{3,0}$ is an isomorphism. It is then necessary to know the arrows $d^3_{3,2q}$; in this *particular case*, a *specific tool* gives the solution; examining the multiplicative structure of the analogous spectral sequence in cohomology, it can be proved the arrow $d^3_{3,2q} : \mathbb{Z} \to \mathbb{Z}$ is the multiplication by $q+1$. This implies the $E^3_{3,2q}$ die and $E^r_{0,2q} = \mathbb{Z}_q$ for $4 \leq r \leq \infty$ and $q \geq 2$. So that the Serre spectral sequence entirely gives the homology groups $H_0(X_4) = \mathbb{Z}$, $H_{2n}(X_4) = \mathbb{Z}_n$ for $n \geq 2$ and the other $H_n(X_4)$ are null. In particular, please believe the Hurewicz theorem [66, [redacted]] and the long exact sequence of homotopy [66, [redacted]] imply $\pi_4(S^3) = \pi_4(X_4) = H_4(X_4) = \mathbb{Z}_2$, a result known by Freudenthal.

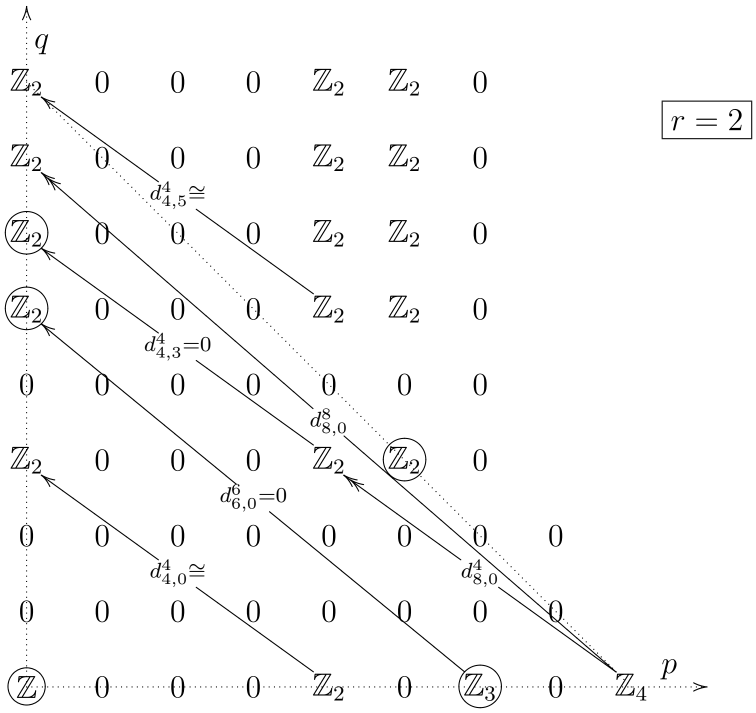
Conclusion: the computation of $H_*(X_4)$ needs more information than which is given by the spectral sequence itself, coming from the multiplicative structure of the cohomology.

[12] Up to sign.

To compute $\pi_5(S^3)$, we must consider a new fibration:

$$F_3 \hookrightarrow X_5 \to X_4$$

where $F_3 = K(\mathbb{Z}_2, 3)$ again is an Eilenberg-MacLane space, with every homotopy group null except $\pi_3(F_3) = \mathbb{Z}_2$, chosen because $\pi_4(X_4) = \mathbb{Z}_2$. We cannot give the details allowing to use the spectral sequence in this case, but the next figure gives an idea of the complexity of the situation[13].



We show the page $r = 2$ and all the arrows which are necessary to determine the $E^\infty_{p,q}$ for $p+q \leq 8$. Up to $p+q \leq 8$, the $E^2_{p,q}$ which remain definitively alive are circled, the others die, and in particular $E^2_{8,0}$ will die in two steps at times $r = 4$ and 8.

The twisting operator of the fibration is the unique one giving $d^4_{4,0} = \mathrm{id}_{\mathbb{Z}_2}$ and $H_3(X_5) = H_4(X_5) = 0$. No choice for $d^6_{6,0}$, it is necessarily the null map, so that $E^7_{0,5} = E^\infty_{0,5} = H_{0,5}(X_5) = H_5(X_5) = \mathbb{Z}_2$. Again the Hurewitz theorem and the long homotopy exact sequence imply $H_5(X_5) = \pi_5(X_5) = \pi_5(X_4) = \pi_5(S^3) = \mathbb{Z}_2$; it was the first important result obtained by Serre.

The arrow $d^4_{8,0}$ is the only non-null arrow from $\mathbb{Z}_4$ to $\mathbb{Z}_2$ and we cannot explain why; this implies the next arrow $d^4_{4,3}$ is null. It was the last possible event for $E^r_{0,6}$, so that $\mathbb{Z}_2 = E^7_{0,6} = E^\infty_{0,6} = H_{0,6}$. Another ingredient for $H_6(X_5)$ is $E^2_{6,0} = E^\infty_{6,0} = \mathbb{Z}_3$. Therefore two stages in the filtration of $H_6(X_5)$ at the abutment, which gives the short exact sequence:

$$0 \leftarrow \mathbb{Z}_3 \leftarrow H_6(X_5) \leftarrow \mathbb{Z}_2 \leftarrow 0$$

[13]The details of this spectral sequence which are shown here have been obtained thanks to Ana Romero's program [50], a good illustration of its possibilities.

The group $H_6(X_5)$ is an extension of $\mathbb{Z}_3$ by $\mathbb{Z}_2$, and fortunately there exists a unique extension $H_6(X_5) = \mathbb{Z}_6$.

Please believe that $d^8_{8,0}$ kills $E^8_{0,7}$ and $E^8_{8,0} = \ker d^4_{8,0} = \mathbb{Z}_2$; in particular $H_7(X_5) = 0$. Also $d^4_{4,5}$ in particular kills $E^4_{0,8}$ which imples $H_8(X_5) = E^\infty_{5,3} = E^2_{5,3} = \mathbb{Z}_2$.

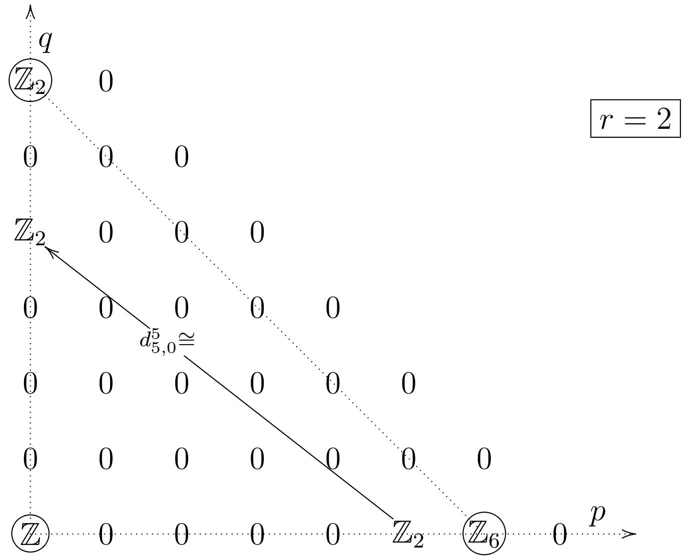
We have obtained the sequence $(\mathbb{Z}, 0, 0, 0, 0, \mathbb{Z}_2, \mathbb{Z}_6, 0, \mathbb{Z}_2)$ for the first homology groups of X_5.

Jean-Pierre Serre was able to obtain all the necessary ingredients for the various $d^r_{p,q}$ which play an essential role in the beginning of this spectral sequence. The main ingredients are the multiplicative structure in cohomology and more generally the module structure with respect to the Steenrod algebra $\mathcal{A}_2$, a subject not studied here.

Let us study now the next fibration:

$$F_4 \hookrightarrow X_6 \to X_5$$

with $F_4 = K(\mathbb{Z}_2, 4)$ and the twisting operator is chosen to have $\pi_5(X_6) = 0$. The part of the spectral sequence interesting for us is simple, we need only the part $p + q \leq 6$:



The same argument as before produces a short exact sequence:

$$0 \leftarrow \mathbb{Z}_6 \leftarrow H_6(X_6) \leftarrow \mathbb{Z}_2 \leftarrow 0$$

but this time two possible extensions, the trivial one $\mathbb{Z}_2 + \mathbb{Z}_6$ and the twisted one $\mathbb{Z}_{12}$. And the Serre spectral sequence does not give any information, given the available data, which allows us to choose the right extension. The conclusion of Serre was only: "The group $\pi_6(S^3) = \pi_6(X_6) = H_6(X_6)$ has 12 elements". Two years later, Barratt and Paechter, using a *quite specific method*, proved the group $\pi_6(S^3)$ in fact contains an element of order 4, so that finally $\pi_6(S^3) = \mathbb{Z}_{12}$, it is the non-trivial extension which is the right one. See [5] and also [58, pp.105-110].

The modern process to determine homotopy groups consists in using the Adams spectral sequence and the numerous other related spectral sequences. Some exact sequences, in particular the chromatic exact sequence, are also very useful. The basic reference about these methods is the marvelous book [49]. It is a marvelous book, numerous important and spectacular results are obtained, but no spectral sequence in this book is made *constructive*.

4 Effective homology.

4.1 Notion of *constructive* mathematics.

Standard mathematics are based on Zermelo-Fraenkel (ZF) axiomatics. When *existence* results are involved, another axiomatics, the *constructive* logic, allows the user to express the results in a more precise way; in this constructive context, one carefully distinguishes the situation where some existence result should... exist (!) from the other situation where a *constructive* process is exhibited producing a copy of the object the existence of which is stated.

The most common example allowing a novice to understand the difference is the following. Question: does there exist two irrational real numbers α and β such that α^β is rational? Let us inspect $\gamma = \sqrt{2}^{\sqrt{2}}$. If γ is rational, then $\alpha = \beta = \sqrt{2}$ is a solution. Otherwise γ is irrational, but then $\alpha = \gamma$ and $\beta = \sqrt{2}$ is a solution, for $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$ is rational. This solution is correct in ZF, but is not in constructive logic. The point is the following; in the existence statement:

$$(1) \qquad (\exists \alpha \in \mathbb{R} - \mathbb{Q})(\exists \beta \in \mathbb{R} - \mathbb{Q})(\alpha^\beta \in \mathbb{Q})$$

you *did not* give a process allowing the user to *construct* such a pair (α, β). You have only produced two candidate solutions $(\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}^{\sqrt{2}}, \sqrt{2})$ and an argument explaining that one of both candidate solutions must satisfy the required property; but which one, this remains *unknown*: you are not able to produce *one* genuine solution.

In constructive logic, if P is a predicate, $\neg\neg P$[14] is not equivalent to P. In the above example, we have only proved:

$$(2) \qquad \neg\neg(\exists \alpha \in \mathbb{R} - \mathbb{Q})(\exists \beta \in \mathbb{R} - \mathbb{Q})(\alpha^\beta \in \mathbb{Q})$$

Let us detail this point. The precise interpretation of $\neg P$ is $P \Rightarrow \perp$ to be read: P implies a contradiction. Typically, $\neg(\sqrt{2} \in \mathbb{Q})$, because if some rational p/q is a square root of 2, Euclid's analysis of the prime decompositions of p and q generates a contradiction. Proving the double negation (2) consists in proving the statement:

$$(3) \qquad (\exists \alpha \in \mathbb{R} - \mathbb{Q})(\exists \beta \in \mathbb{R} - \mathbb{Q})(\alpha^\beta \in \mathbb{Q}) \Rightarrow \perp$$

[14] $\neg$ = not.

implies a contradiction, that is:

$$(4) \qquad ((\exists \alpha \in \mathbb{R} - \mathbb{Q})(\exists \beta \in \mathbb{R} - \mathbb{Q})(\alpha^\beta \in \mathbb{Q}) \Rightarrow \bot) \Rightarrow \bot$$

Let us assume this statement (3). We *then* prove firstly $\sqrt{2}^{\sqrt{2}} \in \mathbb{R} - \mathbb{Q}$. In fact applying (3) to $\alpha = \beta = \sqrt{2}$ known irrational (Euclid), the hypothesis $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ generates a contradiction, which is the *very definition* of $\sqrt{2}^{\sqrt{2}} \in \mathbb{R} - \mathbb{Q}$. We can again apply (3) this time to $\alpha = \sqrt{2}^{\sqrt{2}}$, now known $\in \mathbb{R} - \mathbb{Q}$, and $\beta = \sqrt{2}$; the computation $\alpha^\beta = 2$ proves $\alpha^\beta \in \mathbb{Q}$, so that we have proved (3) implies a contradiction; in other words we have proved (4), that is, (2).

On the contrary, our discussion is not a *constructive* proof of (1), so that (1) and (2) are not equivalent. Mathematicians usually think "not-not = yes", but if the existence is constructively understood, you see $\neg\neg P$ is not necessarily equivalent to P. You see also the constructive interpretation of (2) gives a better interpretation of the ZF statement (1): constructive mathematics is more precise and richer than ZF mathematics, and mainly closer to the actual world. Consider these statements:

(1) It is false there is no book about constructive analysis in this library.

(2) The upper shelf to the left of the east window at the second floor of the library has a book about constructive analysis[15].

Are these statements equivalent?

A constructive interpretation of existence quantifiers is an elegant way to implicitly require as far as possible algorithms producing the objects whose existence is stated. Sometimes it is possible, sometimes not; sometimes the problem is open.

To be complete about our example around the $\sqrt{2}$'s, we must mention that in fact a famous theorem of Gelfond and Schneider proves a^b is transcendant as soon as a and b are algebraic, $a \neq 0, 1$ and $b \in \mathbb{R} - \mathbb{Q}$. So that $\sqrt{2}^{\sqrt{2}}$ is transcendant and $(\alpha, \beta) = (\sqrt{2}^{\sqrt{2}}, \sqrt{2})$ is this time, thanks to Gelfond and Schneider, a constructive solution of our problem. But the proof is a long story!

Another *constructive* solution[16] is quite elementary; it can be obtained as follows: take $\alpha = \sqrt{2}$ and $\beta = 2\log_2 3$; Euclid knew $\alpha \notin \mathbb{Q}$ and if he had known the definition of $\log_2$, it would have been able to prove $\beta \notin \mathbb{Q}$ as well. And $\alpha^\beta = 3$.

4.2 Existential quantifiers and homological algebra.

Let C_* be a chain-complex and n be some integer. Let us study the statement $H_n(C_*) = 0$. By definition $H_n(C_*) = Z_n(C_*)/B_n(C_*)$, and $H_n(C_*) = 0$ means any

[15]Maybe the famous book by Bishop and Bridges, Springer-Verlag, 1985.
[16]Communicated by Thierry Coquand.

n-cycle is an n-boundary. In a still more detailed way:

$$(\forall c \in C_n)((dc = 0) \Rightarrow ((\exists c' \in C_{n+1})(dc' = c)))$$

And the critical question is the following: what about the exact status of the existential quantifier?

In ordinary homological algebra, no constructiveness property is required for this quantifier and, because constructing this preimage c' is most often a little difficult, standard homological algebra is in a sense a catalog of methods allowing you to prove some homology group is null without exhibiting an algorithm constructing a boundary preimage for a cycle. For example if you can insert your homology group in an exact sequence where both close groups are null, then you know your group is null too, and in ordinary homological algebra, this is enough.

But this habit has a severe drawback. For example we have explained how Jean-Pierre Serre was unable to choose between $\mathbb{Z}_2 + \mathbb{Z}_6$ and $\mathbb{Z}_{12}$ when computing the group $\pi_6 S^3$. The homology groups $E^2_{6,0}$ and $E^2_{0,6}$ of his spectral sequence did not give any information about the nature, trivial or not, of the extension of $\mathbb{Z}_6$ by $\mathbb{Z}_2$. We will give later an analysis of this difficulty: it comes from a lack of representants of homology classes. When it is claimed $H_2(C_*) = \mathbb{Z}_6$, it is in fact an unfortunate shorthand for: there *exists* an isomorphism $H_2(C_*) \underset{\phi}{\overset{\psi}{\rightleftarrows}} \mathbb{Z}_6$; but this claimed existence most often is not constructive. To make it constructive, you must be able to *construct* ψ, in other words you must be able to *construct* the homology classes in front of the elements of $\mathbb{Z}_6$, for example by exhibiting cycles $(z_i)_{0 \leq i \leq 5}$ representing them. It is not finished, you must next construct ϕ; let $\mathfrak{h} \in H_2(C_*)$; most often the homology class $\mathfrak{h}$ is given through a cycle z, and because ψ is assumed available, defining $\phi(\mathfrak{h})$ amounts to identify which z_i is homologous to z. Let us assume in a particular case it is z_5; this means $(\exists c \in C_3)(dc = z - z_5)$, again an existential quantifier.

We will explain how it is possible, and elementary, to systematically organize homological algebra in a constructive style. It is not hard and very useful. The fuzzy classical tools such as exact and spectral sequences will easily so become *algorithms* allowing you to compute wished homology and homotopy groups. Of course you must remain lucid about the complexity of the algorithms so obtained, but there is an interesting intermediate work level where these algorithms will produce results otherwise unreachable.

4.3 The homological problem for a chain-complex.

We translate the constructiveness requirement roughly described in the previous section into a definition. This definition a little heavy but unavoidable is essentially *temporary*. It will be soon replaced by the notion of *reduction*.

Definition 41 — Let $\mathfrak{R}$ be a ground ring and C_* a chain-complex of $\mathfrak{R}$-modules. A *solution S of the homological problem for C_** is a set $S = (\sigma_i)_{1 \leq i \leq 5}$ of five *algorithms*:

1. $\sigma_1 : C_* \to \{\perp, \top\}$ ($\perp$ = false, $\top$ = true) is a predicate deciding for every $n \in \mathbb{Z}$ and every n-chain $c \in C_n$ whether c is an n-cycle or not, in other words whether $dc = 0$ or $dc \neq 0$, whether $c \in Z_n(C_*)$ or not.

2. $\sigma_2 : \mathbb{Z} \to \{\mathfrak{R}\text{-modules}\}$ associates to every integer n some $\mathfrak{R}$-module $\sigma_2(n)$ in principle isomorphic to $H_n(C_*)$. The image $\sigma_2(n)$ will *model* the isomorphism class of $H_n(C_*)$ in an effective way to be defined.

3. The algorithm σ_3 is indexed by $n \in \mathbb{Z}$; for every $n \in \mathbb{Z}$, the algorithm $\sigma_{3,n} : \sigma_2(n) \to Z_n(C_*)$ associates to every n-homology class $\mathfrak{h}$ coded as an element $\mathfrak{h} \in \sigma_2(n)$ a cycle $\sigma_{3,n}(\mathfrak{h}) \in Z_n(C_*)$ *representing* this homology class.

4. The algorithm σ_4 is indexed by $n \in \mathbb{Z}$; for every $n \in \mathbb{Z}$, the algorithm $\sigma_{4,n} : C_n \supset Z_n(C_*) \to \sigma_2(n)$ associates to every n-cycle $z \in Z_n(C_*)$ the homology class of z coded as an element of $\sigma_2(n)$.

5. The algorithm σ_5 is indexed by $n \in \mathbb{Z}$; for every $n \in \mathbb{Z}$, the algorithm $\sigma_{5,n} : ZZ_n(C_*) \to C_{n+1}$ associates to every n-cycle $z \in Z_n(C_*)$ *known as a boundary* by the previous algorithm, a boundary preimage $c \in C_{n+1}$: $dc = z$. In particular $ZZ_n(C_*) := \ker \sigma_{4,n}$

Several complements are necessary to clarify this definition.

The computational context needs some method to *code* on our theoretical or concrete machine the chain-complex C_* and the homology groups $H_n(C_*)$; and also their elements. We will see a *locally effective* representation of C_* will be enough; this subtle notion, very important, in fact most often ordinarily underlying, is detailed in the next section.

In most important cases, the set of interesting isomorphism classes of $\mathfrak{R}$-modules is countable, and some simple process defines a relevant isomorphism class as a finite machine object. If $\mathfrak{R}$ is a principal ring, $\mathbb{Z}$ for example, an $\mathfrak{R}$-module *of finite type* H may be described as a sequence $H = (d_1, \ldots, d_r) \in \mathfrak{R}^r$ for some r, the pseudo-rank, the sequence H satisfying the *divisor condition*: d_1 divides d_2, which divides d_3 and so on up to d_r. For example the $\mathbb{Z}$-module $\mathbb{Z}^2 + \mathbb{Z}_6 + \mathbb{Z}_{15}$ would be represented as the sequence $(3, 30, 0, 0)$. This representation is perfect: the correspondance between isomorphism classes and representations is bijective[17]. An element of such an $\mathfrak{R}$-module H is then coded as a simple machine object using the standard structured types.

As usual, an isomorphism class is defined through a *representant* of this class, but to make complete such a representation, an isomorphism must also *effectively* be given between the original group and the representant of the isomorphism class: this is the role of σ_3 and σ_4. In our context, $\sigma_{3,n}$ describes the isomorphism from the *representant* $\sigma_2(n)$ of the homology group to the *genuine* homology group $H_n(C_*)$, an element of the last group being in turn represented by a cycle. The algorithm $\sigma_{4,n}$ is the reciprocal. Note the map $\sigma_{3,n}$ cannot be in general a module morphism.

[17]In a different context, the presentation of a group by finite sets of generators and relators is not perfect: no *effective* canonical presentation, because of the Gödel-Novikov-Rabin theorem.

In the chain-complex $0 \leftarrow \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \leftarrow 0$ null outside degrees 0 and 1, $Z_0(C_*) = \mathbb{Z}$ and $H_0(C_*) = \mathbb{Z}_2$. The map $\sigma_{4,0}$ is surjective and a morphism, but the map $\sigma_{3,0}$, a section of the previous one, cannot be a module morphism. This unpleasant possible behavior will soon be avoided thanks to the notion of *reduction*.

Observe a homology class is represented in two different ways, and it is important to understand the subtle difference. An "actual" n-homology class is represented by a cycle $z \in Z_n(C_*)$, while its image $\sigma_{4,n}(z)$ represents the same element in the model $\sigma_2(n)$ of the isomorphism class of the homology group.

The algorithm σ_5 is in particular a *certificate* for the claimed properties of σ_3 and σ_4, but its role is not at all limited to this authentication. We will see it is the main ingredient allowing us to make constructive the usual exact and spectral sequences.

4.4 Notion of locally effective object.

When you use a simple pocket computer, this computer is able to compute for example the sum of two integers a and b for a large set of integers $\mathbb{Z}' \subset \mathbb{Z}$. This situation is quite common, but not precisely enough analyzed. We will describe this situation by a convenient terminology; we will say the computer contains a *locally effective* version of the standard ring $\mathbb{Z}$.

The mathematical ring $\mathbb{Z}$ is a large set provided with a few operators. On your computer, you can ask for $2 + 3$ and the answer is 5. You *enter* (input) two particular elements of $\mathbb{Z}$ and another one has been computed, the right terminology being: '5' has been *returned* (output). Any analogous computation can be done, at least if it is possible to enter the arguments, when they are not too large. Note no *global* description of $\mathbb{Z}$ is given by your computer. But for *arbitrary* integers a and b, the computer can effectively compute $a + b$. We will use in such a situation the following expression: the addition on $\mathbb{Z}$ is *locally effective*; this expression is a little inappropriate, no topology here to justify the adverb "locally", but experience shows it is very convenient. In a detailed way, we mean there is no *global* implementation of the addition; the possible *global* properties of the addition, for example associativity, commutativity, are unreachable by your computer, but this does not prevent you from using it fruitfully. It is not frequent to need a global property of the addition, most often we use only "local", more precisely *elementwise*, properties. For example for the specific elements 2 and 3, the sum is 5.

For two *arbitrary* elements $a, b \in \mathbb{Z}$, the computer can compute $a + b$; really arbitrary? Not exactly. Not many computers could accept for example $a = 10^{10^{10}}$. The user of such a concrete locally effective implementation of $\mathbb{Z}$ usually knows he must be sensible about input size. The specific problem met here most often is a problem of memory size, or technical bounds. In computational algebra systems allowing you to handle the so-called *extended* integers, with a claimed arbitrary number of digits, you are yet limited by the memory size of your machine. For a specific computation you could after all buy more memory to succeed[18]. On pocket computers, technical limits are most often given, maybe you are limited to integers with less than ten decimal digits. From a theoretical mathematical point of view, these constraints are most often neglected, without any serious drawback, at least in a first step. The underlying implicit statement is: when you will use *concrete* implementations of locally effective objects, be careful, you can meet memory limitations, otherwise the results will be correct. And this is enough in a first step. Of course, time and space complexity is an important subject, theoretically as much as practically, but we decide it is another subject, which of course will be quickly present in concrete calculations.

Another point is to be considered. If you try to enter the "arbitrary integer" `234hello567`, we hope your computer or computational algebra system complains! Another formalism is here necessary. The universe $\mathcal{U}$ is the set of all the *objects* that can be handled[19] by a machine. The set of "legal" integers is a small subset of it; the computer scientists use the notion of *type* to formalize this point. Specific machine objects, more precisely specific machine *predicates*, can be used to verify whether an object is an integer or not. Which allows the machine or the program, when it is safely organized, to detect an incoherent input. Situations are quite different according to concrete implementations. The simplest pocket computers do not have alphabetic keys. More sophisticated ones have and almost always detect our incorrect integer. If you use an intermediary programming language, according to the language, `234hello567` is an object or not: in Lisp yes, in C not. In Lisp this object is accepted but it is a symbol, which cannot be a legal argument for addition, a type error is in principle detected. In C this character string does not denote any machine object and the compiler or more rarely the interpreter will detect an incoherent input, most often being unable to guess what your intention could be.

These technical but anavoidable considerations will be formalized here by characteristic functions. A locally effective object will contain a *membership* predicate, that is an algorithm $\chi : \mathcal{U} \to \{\bot, \top\}$ allowing the user or the program, if necessary, to verify the object it must process has the right type[20], that is, actually is a member of the underlying set.

Another more subtle predicate must also be used. On most simple pocket computers, instead of keying `2`, you could enter as well `0.002E3`, two *different* notations are possible, because $2 = 0.002 \times 10^3$. And this is a permanent problem when implementing mathematical objects: *different* machine objects can *code* the *same* mathematical object. Sometimes it is an extremely technical point: the integer object `2` somewhere in the machine is or is not "equal" to another object again `2` but somewhere else in the machine[21]. Sometimes, such a decision depends

[18] But memory extensions, except for Turing machines, have their own technical bounds!

[19] Without taking account of size limitations! We will not make this precision anymore.

[20] Such a characteristic function is a universal predicate, and an interesting question is to construct the type of the universal predicates, in other words the type of types! If you study a little more this matter, you will quickly rediscover Gödel's incompleteness theorem.

[21] Only Common Lisp correctly handles this matter, see the Lisp functions `eq` and `eql`.

on the technical choice of the user: if you have to implement $\mathbb{Z}_5 := \mathbb{Z}/5\mathbb{Z}$, you can decide to implement an object as an integer `0` or `1` or `2` or `3` or `4`, why not; but sometimes it is much better to decide to represent an element of $\mathbb{Z}_5$ by an arbitrary machine integer, taking care that in fact `12` and `17` represent the same element of $\mathbb{Z}_5$.

We will not give more details about this notion of locally effective object. The numerous examples studied in this text are sufficient illustrations.

4.5 Notion of effective object.

On the contrary, we must sometimes be able to "know everything" about an object, including the *global* properties. For example if you intend to compute some homology group $H_n(C_*)$ of the chain-complex C_*, you must know the *global* nature of C_k for $k = n-1, n, n+1$, and you must know also the differentials d_k and d_{k+1} in such a way you can compute $\ker d_k$, $\operatorname{im} d_{k+1}$ and finally the looked-for homology group.

If the chain-complex is only *locally effective*, these calculations in general are not possible, you must have more information about your chain-complex. We will say a chain-complex is *effective* when every chain group C_n is of finite type. Then the a priori locally effective implementation of a boundary operator d_n becomes effective and the homology group can be computed. Instead of painful abstract definitions, we prefer to illustrate this point by a typical Kenzo example.

Let us assume we are interested by $H_7(K(\mathbb{Z}, 3))$. The Eilenberg-MacLane space $K(Z, 3)$ has the following characteristic property: its homotopy groups are null except $\pi_3 K(\mathbb{Z}, 3) = \mathbb{Z}$. The Kenzo program can construct it:

```
> (setf KZ3 (k-z 3)) ✠
[K11 Abelian-Simplicial-Group]
```

The simplicial set $K(\mathbb{Z}, 3)$ is locally effective, and in principle it is not possible to deduce from its implementation its homology groups. But the Kenzo program is intelligent enough to use the *definition* of $K(\mathbb{Z}, 3)$ to undertake sophisticated computations giving the result. Look at the (Kenzo) *definition* of this object.

```
> (dfnt KZ3) ✠
(CLASSIFYING-SPACE [K6 Abelian-Simplicial-Group])
```

It is the *classifying space* of another simplicial group. Using this definition and others, Kenzo can compute the homology group.

```
> (homology KZ3 7) ✠
Homology in dimension 7 :
Component Z/3Z
---done---
```

But let us play now to *hide* the definition. We reinitialize – `cat-init` – the environment, otherwise it would not be sufficient.

```
> (cat-init) ✠
---done---
> (setf KZ3 (k-z 3)) ✠
[K11 Abelian-Simplicial-Group]
> (setf (slot-value KZ3 'dfnt) '(hidden-definition)) ✠
(HIDDEN-DEFINITION)
> (homology KZ3 7) ✠
Error: I don't know how to determine the effective homology of: [K11
Abelian-Simplicial-Group] (Origin: (HIDDEN-DEFINITION)).
```

This is due to the fact that the chain-complex associated with our $K(\mathbb{Z}, 3)$ is only *locally effective*: no *global* information is reachable:

```
> (basis KZ3 7) ✠
Error: The object [K11 Abelian-Simplicial-Group] is locally-effective.
```

and in fact the basis is infinite. Let us reinstall the right definition:

```
> (setf (slot-value KZ3 'dfnt) `(classifying-space ,(k 6))) ✠
(CLASSIFYING-SPACE [K6 Abelian-Simplicial-Group])
```

The basis of the chain-complex is still unreachable:

```
> (basis KZ3 7) ✠
Error: The object [K11 Abelian-Simplicial-Group] is locally-effective.
```

but the homology group is computable:

```
> (homology KZ3 7) ✠
Homology in dimension 7 :
Component Z/3Z
---done---
```

How this is possible? It is here the *heart* of our subject. Because of the correct definition, Kenzo is able to construct the *effective homology* of $K(\mathbb{Z}, 3)$. Taking account of `efhm` = Effective Homology:

```
> (efhm KZ3) ✠
[K265 Equivalence K11 <= K255 => K251]
```

This homology equivalence is the key point, it is an equivalence between the *locally effective* chain-complex `K11` $= C_*(K(\mathbb{Z},3))$ and the *effective* chain-complex `K251` which cannot be detailed at this point.

```
> (basis (K 11) 7) ✠
Error: The object [K11 Abelian-Simplicial-Group] is locally-effective.
> (basis (K 251) 7) ✠
(<<Abar[7 <<Abar[2 S1][2 S1][2 S1]>>]>>)
```

In fact there is only one generator in C_7(`K251`), which does not prevent the chain-complex `K251` from being homology equivalent to `K11`, the C_7 of which being on the contrary not at all of finite type. And Kenzo, knowing this equivalence, computes in fact the homology group of `K251` when $H_7(K(\mathbb{Z},3))$ is asked for.

The *effective homology theory* is essentially a systematic method combining locally effective chain-complexes with effective chain-complexes through homology equivalences. A locally effective chain-complex is too "vague" to allow us to compute its homology groups, but it is so possible to implement infinite objects such as our Eilenberg-MacLane space $K(\mathbb{Z},3)$. The effective chain-complexes are objects where homology groups can be elementary computed, but only simple objects of finite type can be so implemented. Homology equivalences will allow us to settle *bridges* between both notions, making homological algebra *effective*.

4.6 Reductions.

Definition 41 is relatively complex and the notion of *reduction* is an interesting intermediate organization allowing the topologist to work on the contrary in a convenient environment, from a traditional mathematical point of view and also when computer implementations are planned.

Definition 42 — A *reduction* $\rho : \widehat{C}_* \ggg C_*$ is a diagram:

$$\rho = \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\rightleftarrows}} C_*}$$

where:

1. $\widehat{C}_*$ and C_* are chain-complexes.
2. f and g are chain-complex morphisms.
3. h is a homotopy operator (degree +1).
4. These relations are satisfied:
   (a) $fg = \mathrm{id}_{C_*}$.
   (b) $gf + dh + hd = \mathrm{id}_{\widehat{C}_*}$.
   (c) $fh = hg = hh = 0$.

A reduction is a particular homology equivalence between a *big* chain-complex $\widehat{C}_*$ and a *small* one C_*. This point is detailed in the next proposition.

Proposition 43 — *Let $\rho : \widehat{C}_* \Rrightarrow C_*$ be a reduction. This reduction is equivalent to a decomposition: $\widehat{C}_* = A_* \oplus B_* \oplus C'_*$:*

1. *$\widehat{C}_* \supset C'_* = \operatorname{im} g$ is a subcomplex of $\widehat{C}_*$.*
2. *$A_* \oplus B_* = \ker f$ is a subcomplex of $\widehat{C}_*$.*
3. *$\widehat{C}_* \supset A_* = \ker f \cap \ker h$ is not in general a subcomplex of $\widehat{C}_*$.*
4. *$\widehat{C}_* \supset B_* = \ker f \cap \ker d$ is a subcomplex of $\widehat{C}_*$ with null differentials.*
5. *The chain-complex morphisms f and g are inverse isomorphisms between C'_* and C_*.*
6. *The arrows d and h are module isomorphisms of respective degrees -1 and +1 between A_* and B_*.*

In other words a reduction is a compact and convenient form of the following diagram.

$$\begin{tikzcd}
\{ & \cdots \arrow[r, shift left, "h"] & \widehat{C}_{n-1} \arrow[l, shift left, "d"] \arrow[r, shift left, "h"] \arrow[d, equal] & \widehat{C}_n \arrow[l, shift left, "d"] \arrow[r, shift left, "h"] \arrow[d, equal] & \widehat{C}_{n+1} \arrow[l, shift left, "d"] \arrow[r, shift left, "h"] \arrow[d, equal] & \cdots \arrow[l, shift left, "d"] & \} = & \widehat{C}_* \arrow[d, equal] \\
\{ & \cdots & \overbrace{A_{n-1}} \arrow[ddl, shift left, "d", "\cong"'] \arrow[d, dotted, dash] & \overbrace{A_n} \arrow[ddl, shift left, "d", "\cong"'] \arrow[d, dotted, dash] & \overbrace{A_{n+1}} \arrow[ddl, shift left, "d", "\cong"'] \arrow[d, dotted, dash] & \cdots \arrow[ddl, shift left, "d", "\cong"'] & \} = & \overbrace{A_*} \arrow[d, dotted, dash] \\
& & \oplus \arrow[d, dotted, dash] & \oplus \arrow[d, dotted, dash] & \oplus \arrow[d, dotted, dash] & & & \oplus \arrow[d, dotted, dash] \\
\{ & \cdots \arrow[uur, shift left, "h"] & B_{n-1} \arrow[uur, shift left, "h"] \arrow[d, dotted, dash] & B_n \arrow[uur, shift left, "h"] \arrow[d, dotted, dash] & B_{n+1} \arrow[uur, shift left, "h"] \arrow[d, dotted, dash] & \cdots & \} = & B_* \arrow[d, dotted, dash] \\
& & \oplus \arrow[d, dotted, dash] & \oplus \arrow[d, dotted, dash] & \oplus \arrow[d, dotted, dash] & & & \oplus \arrow[d, dotted, dash] \\
\{ & \cdots & \underbrace{C'_{n-1}} \arrow[l, "d"'] \arrow[d, bend right, "f"'] \arrow[d, phantom, "\cong"] & \underbrace{C'_n} \arrow[l, "d"'] \arrow[d, bend right, "f"'] \arrow[d, phantom, "\cong"] & \underbrace{C'_{n+1}} \arrow[l, "d"'] \arrow[d, bend right, "f"'] \arrow[d, phantom, "\cong"] & \cdots \arrow[l, "d"'] & \} = & \underbrace{C'_*} \arrow[d, bend right, "f"'] \arrow[d, phantom, "\cong"] \\
\{ & \cdots & C_{n-1} \arrow[l, "d"'] \arrow[u, bend right, "g"'] & C_n \arrow[l, "d"'] \arrow[u, bend right, "g"'] & C_{n+1} \arrow[l, "d"'] \arrow[u, bend right, "g"'] & \cdots \arrow[l, "d"'] & \} = & C_* \arrow[u, bend right, "g"']
\end{tikzcd}$$

It is a simple exercise of elementary linear algebra to prove the equivalence between the above diagram and the initial reduction. Every chain group $\widehat{C}_n$ is then decomposed into three components, A_n made of chains in canonical bijection with B_{n-1} thanks to d and h. We can consider A_n is a collection of n-chains ready to explain the elements of B_{n-1} are not only cycles, but also boundaries. B_n is a collection of cycles known as boundaries, because of the bijection between A_{n+1}

and B_n again through d and h. Finally the component C'_n is a copy of C_n and their homological natures therefore are the same.

A reduction $\rho : \widehat{C}_* \Rightarrow\!\!\!\!\!\Rightarrow C_*$ is a decomposition $\widehat{C}_* = \ker f \oplus C'_*$ in two components; no specific information about the second one other than $C'_* \cong C_*$; but the first one $\ker f$ is *acyclic*, for the restriction of the relation $\mathrm{id}_{\widehat{C}_*} = gf + dh + hd$ to $\ker f$ is simply $\mathrm{id}_{\ker f} = dh + hd$; note in particular $\ker f$ is a subcomplex of $\widehat{C}_*$: f is a chain-complex morphism, that is, $df = fd$, which implies $d(\ker f) \subset \ker f$. Note also $h(\widehat{C}_*) \subset \ker f$, a consequence of $fh = 0$. The component $\ker f$, known as acyclic, is in turn decomposed in two components, $\ker f = A_* + B_*$ with $A_* = \ker f \cap \ker h$ and $B_* = \ker f \cap \ker d$. This can be considered as a *Hodge decomposition* of $\widehat{C}_*$, describing in a detailed way why the homology groups of $\widehat{C}_*$ and C_* are canonically isomorphic.

Theorem 44 — *Let $\rho = (f, g, h) : \widehat{C}_* \Rightarrow\!\!\!\!\!\Rightarrow C_*$ be a reduction where the chain-complexes $\widehat{C}_*$ and C_* are locally effective. If the homological problem is solved in the small chain-complex C_*, then the reduction ρ induces a solution of the homological problem for the big chain-complex $\widehat{C}_*$.*

Proof. Let us examine the criteria of Definition 41.

1. Let $c \in \widehat{C}_*$; the chain-complex $\widehat{C}_*$ is locally effective and the "local" calculation dc can be achieved, which allows you to determine whether the chain c satisfies $dc = 0$ or not, whether c is a cycle or not.

2. The known relations $\mathrm{id}_{C_*} = fg$ and $\mathrm{id}_{\widehat{C}_*} = gf + dh + hd$ imply f and g are inverse homology equivalences. The homology groups $H_n(\widehat{C}_*)$ and $H_n(C_*)$ are *canonically* isomorphic. Let σ_* be the algorithms provided by the solution of the homological problem for C_* and let us call $\widehat{\sigma}_*$ the algorithms to be constructed for $\widehat{C}_*$. We can choose in particular $\widehat{\sigma}_{2,n} = \sigma_{2,n}$, the last *equality* being a *genuine* one.

3. The chain morphism f induces an isomorphism between $H_n(\widehat{C}_*)$ and $H_n(C_*)$. This allows us to choose $\widehat{\sigma}_{3,n}(z) := \sigma_{3,n}(f(z))$.

4. In the same way, choose $\widehat{\sigma}_{4,n}(\mathfrak{h}) := g(\sigma_{4,n}(\mathfrak{h}))$.

5. Finally, if $z \in \widehat{C}_n$ is a cycle known homologous to zero, a boundary preimage is $\widehat{\sigma}_{5,n}(z) := h(z) + g(\sigma_{5,n}(f(z)))$. In fact: $d(hz + g(\sigma_{5,n}(f(z)))) = dhz + gd\sigma_{5,n}(f(z)) = dhz + gfz = z - hdz = z$, for g is a chain-complex morphism, $\sigma_{5,n}$ finds boundary preimages, and z is a cycle. ∎

Corollary 45 — *If $\rho = (f, g, h) : \widehat{C}_* \to C_*$ is a reduction where $\widehat{C}_*$ is* locally effective *and C_* is* effective*, then this reduction produces a solution of the homological problem for $\widehat{C}_*$.*

Proof. The small chain-complex C_* is effective and a solution of the homological problem for C_* therefore is elementary. ∎

Proposition 46 — *Let $\rho = (f, g, h) : \widehat{C}_* \to C_*$ be a reduction, where the homological problem is solved for $\widehat{C}_*$. Then the homological problem is also solved for the small chain complex C_*.*

Proof. The small chain complex being a sub-chain-complex of the big one, the situation is more comfortable. The only point deserving a little attention is the search of a boundary preimage for a C_*-cycle known being a boundary: exercise. ∎

Corollary 47 — *If $\varepsilon : C_* \lll\!\!\Longrightarrow\!\!\ggg C'_*$ is an equivalence between two chain-complexes, a solution of the homological problem for C'_* gives a solution of the same problem for C_*. In particular, if C'_* is effective, the homological problem is solved for C_*.*

The reader probably wonders why, in presence of such a reduction $\rho : \widehat{C}_* \Rrightarrow C_*$, the user continues to give some interest to $\widehat{C}_*$. The big chain-complex $\widehat{C}_*$ is the direct sum of the small one C_* and $\ker f$, the last component not playing any role from a homological point of view. The point is the following: frequently we have to work with chain-complexes which carry more structure than a chain-complex structure. For example if the chain-complex comes from a simplicial set or complex, there is another structure, the simplicial structure which is present, and the chain-complex structure in this case is *underlying*; and it is frequent the chain-complex structure can be *reduced* but the simplicial structure *not*. So that you must continue to play with the big chain-complex $\widehat{C}_*$ and its further simplicial structure, but when the subject is homology, you can transfer the work to the small chain-complex C_*. And the planned work continually is of this sort: playing simultaneously with big objects provided with sophisticated structures, most often not significantly *reducible*, and their small homological reductions.

4.7 Kenzo example.

We want to *concretely* illustrate how reductions between locally effective and effective chain-complexes allow a user to obtain and use the corresponding solution of a homological problem.

The mathematical underlying theory will be explained later in Section 6 and we use here Example 6.8.4 of this section. We consider the polynomial ring $\mathfrak{R} = \mathbb{Q}[t, x, y, z]_0$ and in this ring the ideal:

$$I = <t^5 - x, t^3y - x^2, t^2y^2 - xz, t^3z - y^2, t^2x - y, tx^2 - z, x^3 - ty^2, y^3 - x^2z, xy - tz> .$$

It happens the homology of the *Koszul complex* $\mathrm{Ksz}(\mathfrak{R}/I)$ reflects deep properties of the ideal I. The Koszul complex is a $\mathbb{Q}$-vector space of infinite dimension, but yet an algorithm can compute its *effective* homology. Kenzo constructs the ideal as a list of generators, each generator being a combination (`cmbn`) of monomials, each monomial being a list of exponents, for example `(3 0 1 0)` codes t^3y.

```
> (setf ideal
    (list
     (cmbn 0 1 '(5 0 0 0) -1 '(0 1 0 0))
     (cmbn 0 1 '(3 0 1 0) -1 '(0 2 0 0))
[... 6 lines deleted ...]
     (cmbn 0 1 '(0 1 1 0) -1 '(1 0 0 1)))) ✠
(
----------------------------------------------------------------------{cmbn 0}
<1 * (5 0 0 0)> <-1 * (0 1 0 0)>
------------------------------------------------------------------------------
[... other lines deleted ...]
)
```

The display is simply the list of generatorss, only the first one is given here. The Koszul complex $\mathrm{Ksz}(\mathfrak{R}/I)$ is then constructed.

```
> (setf ksz (k-complex/gi 4 ideal)) ✠
[K5 Chain-Complex]
```

Kenzo returns `K5`, the Kenzo object `#5`, a chain-complex. The ideal in fact is as well generated by the toric generators $x-t^5$, $y-t^7$, $z-t^{11}$; we will see how the *effective* homology of the Koszul complex can *discover* this fact. Three generators and four variables, the quotient is certainly of infinite $\mathbb{Q}$-dimension. If we ask for the $\mathbb{Q}$-basis of the Koszul complex in degree 2 for example, an error is returned.

```
> (basis ksz 2) ✠
Error: The object [K5 Chain-Complex] is locally-effective.
```

Several procedures in Kenzo can compute the effective homology of `K5`. In particular the procedure `koszul-min-rdct` computes the *minimal* effective homology as a reduction.

```
> (setf mrdct (koszul-min-rdct ideal "H")) ✠
[K778 Reduction K5 => K763]
```

The reduction is assigned to the symbol `mrdct`, a reduction of the chain-complex `K5` over the chain-complex `K763`. You observe several hundreds of Kenzo objects, chain-complexes, morphisms, reductions, equivalences, ..., have been necessary to obtain the result, but this work of automatic writing of programs is very fast, less than half a second for our modest laptop. The small chain-complex `K763` is effective. The Lisp statement `(mapcar ...)` gives the list of $\mathbb{Q}$-dimensions from 0 to 4.

```
> (mapcar
     #'(lambda (i) (length (basis (k 763) i)))
     '(0 1 2 3 4)) ✠
(1 3 3 1 0)
```

Let us look for the first generator in degree 2 and compute its differential.

```
> (first (basis (k 763) 2)) ✠
H-2-1
> (? (k 763) 2 *) ✠
----------------------------------------------------------------------{CMBN 1}
------------------------------------------------------------------------------
```

The generator is the symbol `H-2-1` and its differential is null. The esoteric Lisp statement "`(?  (k 763) 2 *)`" is to be understood as follows: as already observed, "`(k 763)`" returns the Kenzo object `K763`, a chain complex. The functional operator '`?`' makes the differential of this chain complex work in this case on a generator of degree 2, namely '`*`', that is, the last object returned by the Lisp interpreter, the symbol `H-2-1`.

In fact the same behaviour can be observed for the eight basis elements: the differential is the null-morphism of degree -1. This property is characteristic of the *minimal* effective homology of our Koszul complex. So that the elements of the list `(1 3 3 1 0)` are the *Betti* numbers of the Koszul complex. The first `3` informs us for example the minimal number of generators for our ideal is 3, while the ideal was defined with 9 generators.

The chain-complex `K763` is nothing but a model for "the" *homology* of our Koszul complex `K5`. The homology class `h-2-1` is represented by the cycle $g($`h-2-1`$)$ if g is the g-component of the reduction `K778` $=$ `mrdct` $= (f, g, h)$.

```
> (g mrdct 2 'h-2-1) ✠
----------------------------------------------------------------------{CMBN 2}
<-1 * ((0 2 0 0) (1 1 0 0))>
<1 * ((4 0 0 0) (1 0 0 1))>
<-1 * ((0 0 0 0) (0 1 0 1))>
------------------------------------------------------------------------------
```

which cycle would be denoted by $-x^2\, dt.dx + t^4\, dt.dz - dx.dz$ in the standard notation explained Section 5.2. You see not only the homology groups are computed, but representants of homology classes can be exhibited.

Let us play now with cycles and boundary preimages. If we take a random element of the Koszul complex, in general it is not a cycle.

```
> (? ksz 2 '((2 0 0 0) (1 1 0 0)))
----------------------------------------------------------------------{CMBN 1}
<-1 * ((0 0 1 0) (1 0 0 0))>
<1 * ((3 0 0 0) (0 1 0 0))>
-----------------------------------------------------------------------------
```

The differential of $t^2\, dt.dx$ is not null; this object is not a cycle. Now the demonstrator goes for a moment into the wings of his theater and comes back with the object z1. Is it a cycle?

```
> (setf z1
    (cmbn 2
          1 '((1 0 1 9) (1 1 0 0))
         -1 '((0 2 0 0) (1 1 0 0))
         -1 '((1 1 0 9) (1 0 1 0))
          1 '((4 0 0 0) (1 0 0 1))
          1 '((2 0 0 9) (0 1 1 0))
         -2 '((1 1 0 0) (0 1 1 0))
          2 '((2 0 0 0) (0 1 0 1))
         -1 '((0 0 0 0) (0 1 0 1))
         -2 '((0 0 0 0) (0 0 1 1)))) ✠
----------------------------------------------------------------------{CMBN 2}
<1 * ((1 0 1 9) (1 1 0 0))>
[... Lines deleted  ...]
-----------------------------------------------------------------------------
> (? ksz z1) ✠
----------------------------------------------------------------------{CMBN 1}
-----------------------------------------------------------------------------
```

The combination $(tyz^9 - x^2)\, dt.dx - txz^9\, dt.dy + t^4\, dt.dz + (t^2z^9 - 2tx)\, dx.dy + (2t^2 - 1)\, dx.dz - 2\, dy.dz$ is a cycle of degree 2. What about its homology class?

```
> (f mrdct z1) ✠
----------------------------------------------------------------------{CMBN 2}
<1 * H-2-1>
<-2 * H-2-3>
-----------------------------------------------------------------------------
```

We obtain the homology class by applying the f-component of the reduction to the cycle; the homology class is `h-2-1` $- 2$ `h-2-3`. The demonstrator again goes into the wings and comes back with another cycle `z2`.

```
> (setf z2
    (cmbn 2
          1 '((1 0 1 9) (1 1 0 0))
         -1 '((1 1 0 9) (1 0 1 0))
          1 '((2 0 0 9) (0 1 1 0)))) ✠
-------------------------------------------------------------------{CMBN 2}
<1 * ((1 0 1 9) (1 1 0 0))>
[... 2 lines deleted ...]
---------------------------------------------------------------------------
> (? ksz z2) ✠

-------------------------------------------------------------------{CMBN 1}
---------------------------------------------------------------------------
> (f mrdct z2) ✠
-------------------------------------------------------------------{CMBN 2}
---------------------------------------------------------------------------
```

This time the cycle is $tyz^9\, dt.dx - txz^9\, dt.dy + t^2z^9\, dx.dy$, but its homology class is null. To obtain a boundary preimage, because the homology is minimal, it is sufficient to apply the h-component of the reduction.

```
> (h mrdct z2)
-------------------------------------------------------------------{CMBN 3}
<1 * ((1 0 1 8) (1 1 0 1))>
<-1 * ((1 1 0 8) (1 0 1 1))>
<1 * ((2 0 0 8) (0 1 1 1))>
---------------------------------------------------------------------------
```

The claimed preimage is $tyz^8\, dt.dx.dz - txz^8 dt.dy.dz + t^2z^8\, dx.dy.dz$. To verify this claim, we compute the difference between the original `z2` and the boundary of the preimage.

```
> (2cmbn-sbtr (cmpr ksz) z2 (? ksz *)) ✠
-------------------------------------------------------------------{CMBN 2}
---------------------------------------------------------------------------
```

A comparison operator between generators is necessary to compute such a difference, it is the reason why the first argument is the comparison operator (`cmpr`) of the Koszul complex (`ksz`). The result is null, OK!

These small computations illustrate how any homological question in the Koszul complex is *effectively* solved, thanks to the reduction `mrdct`. Even if the chain-complex is not of finite $\mathbb{Q}$-type. There remains to understand how it is possible to construct the critical reduction, more generally the necessary equivalence.

4.8 Homological Perturbation theory.

4.8.1 Presentation.

The most important tool allowing us to efficiently work with reductions is the so-called *basic perturbation lemma*, a "lemma" which would be better called the *fundamental theorem of homological algebra*. We intend to construct and study objects that are in a sense *recursively* constructed, that is, constructed from previous objects already studied. And we need tools to study the new objects using the informations that are known for the previous ones.

Typically, many topological spaces can be described as the total space of a fibration. This total space E is then presented as a twisted product of two other spaces: $E := F \times_\tau B$; the space B (resp. F) is the base space (resp. fibre space) and instead of the ordinary product $F \times B$, some important modification in the construction of the product, following the instructions given by the *twisting function* τ, allows one to construct a different space, for some reason or other. For example in Section 3.3.2 we have constructed X_4 and X_5 as twisted products $X_4 = K(\mathbb{Z}, 2) \times_\tau S^3$ and $X_5 = K(\mathbb{Z}_2, 3) \times_{\tau'} X_4$ where τ and τ' were chosen to "kill" the first non-null homotopy group of S^3 and X_4.

So that the game rule is the following. Given: the homological nature of F and B. Problem: How to determine the *same* information for $E = F \times_\tau B$? In this case, the Eilenber-Zilber theorem gives the homology of the *non-twisted* product $E' = F \times B$; and if an appropriate hypothesis is satisfied for B (simple connectivity), then the basic perturbation lemma allows to consider the twisted product E as a *perturbation* of the non-twisted product E' and to obtain the looked-for homological information for E. This will be our *effective version* of the Serre spectral sequence.

Definition 48 — Let (C_*, d) be a chain-complex. A collection of module morphisms $\delta = (\delta_n : C_n \to C_{n-1})_{n\in\mathbb{Z}}$ is called a *perturbation* of the differential d if the sum $d + \delta$ is also a differential, that is, if $(d + \delta)^2 = 0$.

Such a perturbation produces a *new* chain-complex $(C_*, d + \delta)$ and in general the homological nature of the chain-complex is so deeply… perturbed. Two theorems are available in this area. The first one, called the *easy* perturbation lemma, is trivial but useful. The second one, called the *basic* perturbation lemma (BPL) is not trivial at all: in a sense it gives more information than some spectral sequences, typically the Serre and Eilenberg-Moore spectral sequences. The BPL was discovered by Shih Weishu [61] to overcome some gaps in the Serre spectral sequence, and Ronnie Brown gave the abstract modern form [11].

4.8.2 Easy perturbation lemma.

Proposition 49 — *Let $\rho = (f, g, h) : (\widehat{C}_*, \widehat{d}) \Rrightarrow (C_*, d)$ be a reduction and let $\delta : C_* \to C_{*-1}$ be a perturbation of the differential d of the small chain-complex.*

Then a "new" reduction $\rho = (f, g, h) : (\widehat{C}_*, \widehat{d}+\widehat{\delta}) \Rrightarrow (C_*, d+\delta)$ can be constructed above the perturbed the chain-complex.

PROOF. The differential of the small chain-complex is perturbed, so that a priori the components f and g of the reduction ρ are nomore compatible with the differentials $\widehat{d}$ and $d+\delta$. But the reduction ρ induces a decomposition $\widehat{C}_* = \ker f \oplus C'_*$ where $C'_* = \operatorname{im} g$ is a copy of the small chain-complex C_*; so that it is enough to copy also the perturbation, that is, to introduce the perturbation $\widehat{\delta} = g\delta f$ of $\widehat{d}$. The nature of $\ker f$ is not modified and the previous components f, g and h of the reduction ρ can be let unchanged. This is the reason why the new reduction is not so "new", it is the *same* reduction between *different* chain-complexes! ■

4.8.3 Basic perturbation lemma.

The situation is now dramatically harder: we intend to perturb the differential of the *big* chain-complex of the reduction. In general it is not possible to coherently perturb the differential of the small chain-complex, even in modifying the reduction itself. For example Let $\widehat{C}_*$ the "big" chain-complex where $\widehat{C}_n = 0$ except $\widehat{C}_0 = \widehat{C}_1 = \mathbb{Z}$ and $d_1 = \mathrm{id}_\mathbb{Z}$. This chain-complex is acyclic, which implies there is a reduction $\rho = (0, 0, h) : \widehat{C}_* \Rrightarrow 0$ over the null chain-complex. If you introduce the perturbation $\widehat{\delta}_1 = -\mathrm{id}_\mathbb{Z}$, then the differential becomes null, the chain-complex is nomore acyclic and it is not possible to perturb coherently the differential of the null chain-complex, which differential in fact cannot be actually "perturbed". This simple example shows some further hypothesis is necessary to make possible a coherent perturbation for the small chain-complex and for the reduction.

Theorem 50 (Basic Perturbation Lemma) — *Let $\rho = (f, g, h) : (\widehat{C}_*, \widehat{d}) \Rrightarrow (C_*, d)$ be a reduction and let $\widehat{\delta}$ be a perturbation of the differential $\widehat{d}$ of the big chain-complex. We assume the* nilpotency hypothesis *is satisfied: for every $c \in \widehat{C}_n$, there exists $\nu \in \mathbb{N}$ satisfying $(h\widehat{\delta})^\nu(c) = 0$. Then a perturbation δ can be defined for the differential d and a new reduction $\rho' = (f', g, h') : (\widehat{C}_*, \widehat{d}+\widehat{\delta}) \Rrightarrow (C_*, d+\delta)$ can be constructed.*

The nilpotency hypothesis states the composition $h\widehat{\delta}$ is pointwise nilpotent. Note the differential of the small chain-complex is modified but also the components (f, g, h) of the reduction which become something else (f', g', h'): we will have to perturb these components as well.

Which is magic in the BPL is the fact that a sometimes complicated perturbation of the "big" differential can be accordingly reproduced in the "small" differential; in general it is not possible, unless the nilpotency hypothesis is satisfied.

PROOF. Because of the nilpotency condition, the following series have, for each element which they work on, only a finite number of non-null terms and their sums are defined:

$$\phi = \sum_{i=0}^{\infty} (-1)^i (h\widehat{\delta})^i; \qquad \psi = \sum_{i=0}^{\infty} (-1)^i (\widehat{\delta} h)^i.$$

The operators ϕ and ψ have degree 0 and trivially satisfy a few relations; *these relations are the only ones that are from now on utilized*:

$$\begin{aligned}
&\phi h = h\psi \ ; \\
&\widehat{\delta}\phi = \psi\widehat{\delta} \ ; \\
&\phi = 1 - h\widehat{\delta}\phi = 1 - \phi h\widehat{\delta} = 1 - h\psi\widehat{\delta} \ ; \\
&\psi = 1 - \widehat{\delta} h\psi = 1 - \psi\widehat{\delta} h = 1 - \widehat{\delta}\phi h.
\end{aligned}$$

The reduction $\rho' = (f', g', h') : (\widehat{C}_*, \widehat{d'}) \Rrightarrow (C_*, d')$ to be constructed is then simply defined by:

$\widehat{d'} = \widehat{d} + \widehat{\delta}$ is the new differential of $\widehat{C}_*$;
$d' = d + \delta$ is the new differential of C_* where $\delta = f\widehat{\delta}\phi g = f\psi\widehat{\delta} g$;
$f' = f\psi$;
$g' = \phi g$;
$h' = \phi h = h\psi$.

Lemma 51 — *Let (C_*, d) be a chain-complex and let h be an operator on C_* of degree $+1$, satisfying the relations:*

$hh = 0$;
$hdh = h$.

Then $D = dh + hd$ is a projector which splits the chain-complex C_* into the direct sum of chain-complexes $\ker D \oplus \operatorname{im} D$ where the second one is acyclic. More precisely, if γ is the canonical inclusion $\ker D \to C_*$, then $(\mathrm{id} - D, \gamma, h) : C_* \Rrightarrow \ker D$ is a reduction.

Proof. The operator D is a projector, because of the computation: $D^2 = (dh + hd)^2 = dhdh + hdhd = dh + hd = D$ (because $hh = 0$ and $dd = 0$). The operator D and therefore also $\mathrm{id} - D$ are chain-complex morphisms : $d(dh + hd) = dhd = (dh + hd)d$ (because $dd = 0$). The operator h also commutes with D and therefore preserves $\ker(\mathrm{id} - D)$; it is null on $\ker D$, for $(dh + hd) = 0$ implies $h(dh + hd) = h = 0$. ■

Proof of Theorem continued. In the theorem, the operator h does satisfy these relations with respect to $\widehat{d}$, because $hh = 0$ is explicitly required among the reduction properties and $h\widehat{d}h = (1 - \widehat{d}h - gf)h = h$ (because $hh = 0$ and $fh = 0$). The projection $D = \widehat{d}h + h\widehat{d}$ is also the difference $1 - gf$, and therefore the complementary projection $1 - D$ is the composition gf.

The new homotopy operator h' has been defined by $h' = \phi h = h\psi$. Firstly, we naturally obtain from the definition of h' the definitions of f', g' and δ.

The new operator h' satisfies also the relations $h'h' = 0$ and $h'\widehat{d'}h' = h'$. In fact $h'h' = \phi hh\psi = 0$ and $h'\widehat{d'}h' = \phi h(\widehat{d}+\widehat{\delta})h\psi = \phi h\widehat{d}h\psi + \phi h\widehat{\delta}h\psi = \phi h\psi + \phi h(1-\psi) = \phi h = h'$ (because $\widehat{\delta}h\psi = 1 - \psi$).

We then obtain from the lemma the fact that $D' = \widehat{d'}h' + h'\widehat{d'}$ is a projector; let us denote by $\pi = gf$ the complementary projector of D and $\pi' = 1 - D'$ the complementary projector of D'.

We already know the relations $hh = h'h' = 0$. Furthermore $hh' = hh\psi = 0$ and $h'h = \phi hh = 0$. In fact any composition of an operator of type h with an operator of type π is null. Firstly $\pi h = (1 - \widehat{d}h - h\widehat{d})h = h - h\widehat{d}h = h - h = 0$ and $h\pi = h(1 - \widehat{d}h - h\widehat{d}) = h - h\widehat{d}h = h - h = 0$. Next $\pi h' = \pi h\psi = 0$ and $h'\pi = \phi h\pi = 0$. Then $\pi'h' = h'\pi' = 0$ is proved like $\pi h = h\pi = 0$. Finally $\pi'h = (1 - \widehat{d'}h' - h'\widehat{d'})h$; but $h'h = 0$ and $\widehat{d'} = \widehat{d} + \widehat{\delta}$, therefore $\pi'h = h - \phi h(\widehat{d} + \widehat{\delta})h = h - \phi h\widehat{d}h - \phi h\widehat{\delta}h = h - \phi h - (1 - \phi)h = 0$ (because $h\widehat{d}h = h$ and $\phi h\widehat{\delta} = 1 - \phi$). In the same way $h\pi' = h(1 - \widehat{d'}h' - h'\widehat{d'}) = h - h(\widehat{d}+\widehat{\delta})h\psi = h - h\widehat{d}h\psi - h\widehat{\delta}h\psi = h - h\psi - h(1-\psi) = 0$.

Let us now consider the compositions $\pi\pi'\pi$ and $\pi'\pi\pi'$. Firstly $\pi\pi'\pi = \pi(1 - \widehat{d'}h' - h'\widehat{d'})\pi = \pi^2 = \pi$, because $\pi h' = h'\pi = 0$. In the same way $\pi'\pi\pi' = \pi'(1 - \widehat{d}h - h\widehat{d})\pi' = \pi'^2 = \pi'$. Therefore the operators π and π' are inverse morphisms between the images of π' and π ; they are only homomorphisms *of graded modules*, in general non compatible with the natural differentials of the respective images. But the image of π has a bijective mapping towards the small graded module C_* through f and g, so that a composition provides an isomorphism of graded modules between C_* and the image of π' which allows us to install a new differential on C_* deduced from the differential of $\operatorname{im}\pi'$, restriction of $\widehat{d'} = \widehat{d} + \widehat{\delta}$.

Firstly let us note that $h'g = \phi hg = 0$, and that $fh' = fh\psi = 0$. Taking account of what was explained in the previous paragraph, it is natural to define $g' = \pi'g = (1 - \widehat{d'}h' - h'\widehat{d'})g = g - \phi h\widehat{d}g - \phi h\widehat{\delta}g = -\phi hgd + (1 - \phi h\widehat{\delta})g = \phi g$. Then the “projection” f' will be the composition of the actual projection π' with the composition $f\pi$. But $f\pi = f(1 - \widehat{d}h - h\widehat{d}) = f - f\widehat{d}h - fh\widehat{d} = f - dfh - fh\widehat{d} = f$ and we obtain $f' = f\pi\pi' = f\pi' = f(1 - \widehat{d'}h' - h'\widehat{d'}) = f - f\widehat{d}h\psi - f\widehat{\delta}h\psi = -\widehat{d}fh\psi + f(1 - \widehat{\delta}h\psi) = f\psi$. We have obtained the announced formulas for the desired reduction components f' and g'.

The new differential to be installed on the graded module underlying C remains to be determined. We naturally compute: $d + \delta = f\pi(\widehat{d} + \widehat{\delta})\pi'g = f(\widehat{d} + \widehat{\delta})\pi g = f\widehat{d}\pi'g + f\widehat{\delta}\phi g = f\widehat{d}(1 - \widehat{d'}h' - h'\widehat{d'})g + f\widehat{\delta}\phi g = f\widehat{d}g - f\widehat{d}\widehat{d'}h'g - dfh'\widehat{d'}g + f\widehat{\delta}\phi g = f\widehat{d}g + f\widehat{\delta}\phi g = d + f\widehat{\delta}\phi g = d + f\psi\widehat{\delta}g$; we must therefore choose $\delta = f\widehat{\delta}\phi g = f\psi\widehat{\delta}g$.

The basic perturbation lemma is proved. ∎

4.9 Objects with effective homology.

An object *with effective homology* is a complex object made of a *locally effective* object – the object under study, an *effective* object – namely an effective chain-complex describing the homological nature of the object under study, both objects being connected by an appropriate homology equivalence. Because of the latter, the homological problem for the underlying object is solved.

Definition 52 — A *strong homology equivalence*, in short an *equivalence* $\varepsilon : C_* \lll\!\!\Longrightarrow\!\!\ggg D_*$ between two chain-complexes is a pair of reductions connecting C_* and D_* through a third chain-complex $\widehat{C}_*$:

$$\varepsilon = \boxed{C_* \overset{\rho_\ell}{\lll\!\!\Longleftarrow} \widehat{C}_* \overset{\rho_r}{\Longrightarrow\!\!\ggg} D_*}$$

Because of the fundamental importance of this sort of equivalence, this will be simply called in this text an *equivalence*. If the homological problem is solved for D_*, it is also solved for C_*.

Definition 53 — An *object with effective homology* X is a quadruple $X = (X, C_*X, EC_*, \varepsilon)$ where:

- X is a *locally effective* object, the homological nature of which is under study.
- C_*X is the (locally effective) chain-complex canonically associated with X when the homological nature of X is studied.
- EC_* is an *effective* chain-complex.
- Finally ε is an equivalence $\varepsilon : C_*X \lll\!\!\Longrightarrow\!\!\ggg EC_*$.

Typically the object *under study* X could be an infinite simplicial complex; if it is infinite, we must content ourselves with a *locally effective* implementation. Then C_*X is the chain-complex canonically associated with it (Section 2.2.2); it is not of finite type and it is also implemented as a *locally effective* chain-complex. In many situations, the homology groups of this chain-complex yet are of finite type: so that some *effective* chain-complex can have the right homology groups. The last but not the least, an *equivalence* between the genuine chain-complex associated with our object and our effective chain-complex will play an essential role in the next constructions. In most cases, the *basic perturbation lemma* will be the main tool constructing new equivalences from others already constructed.

A good didactic simple example of object with effective homology, didactic but very useful, is the Koszul complex $\mathrm{Ksz}_{\mathfrak{R}}(\mathfrak{R})$, see Section 5.7. If $\mathfrak{k}$ is the underlying ground field, then the Koszul complex has an infinite $\mathfrak{k}$-dimension; but it is a resolution of $\mathfrak{k}$ and its homology is only $\mathfrak{k}$ in dimension 0, nothing else. The reduction $\mathrm{Ksz}_{\mathfrak{R}}(\mathfrak{R}) \Longrightarrow\!\!\ggg \mathfrak{k}$ which will be constructed is the equivalence component $\varepsilon_{\mathfrak{R}}$. So that the quadruple $(\mathrm{Ksz}_{\mathfrak{R}}(\mathfrak{R}), \mathrm{Ksz}_{\mathfrak{R}}(\mathfrak{R}), \mathfrak{k}, \varepsilon_{\mathfrak{R}})$ is a version *with effective homology* of the Koszul complex. In this case, and this is not seldom, the object under study is the chain-complex itself.

The main result of Effective Homology Theory is the following "meta-theorem".

Meta-Theorem 54 — *Let $X_1, \ldots, X_k$ be a collection of objects and ϕ some "reasonable"* constructor $\phi : (X_1, \ldots, X_k) \mapsto X$. *Then a* version with effective homology ϕ_{EH} *can be obtained, constructing a* version X_{EH} with effective homology *of the result X of the construction when* versions with effective homology *of the X_i's are given:*

$$\phi_{EH} : ((X_1, C_*X_1, EC^X_{1\ *}, \varepsilon_1), \ldots, (X_k, C_*X_k, EC^X_{k\ *}, \varepsilon_k)) \mapsto (X, C_*X, EC^X_*, \varepsilon).$$

The nature of *constructive* homological algebra is now simply defined: please transform the standard theorems of homological algebra into instances of this meta-theorem. The version with effective homology ϕ_{EH} of the constructor ϕ is a collection sometimes sizeable of algorithms constructing algorithm components of the result $X_{EH} = (X, C_*X, EC^X_*, \varepsilon)$ from the algorithm components of the data $X_{i,EH} = (X_i, C_*X_i, EC^X_{i\ *}, \varepsilon_i)$. An *algorithm* constructing *algorithms* from other *algorithms* requires *functional programming*; this wonderful tool is theoretically known since Church's work in logic [15], a theoretical work leading to the currently most complete programming language, Common Lisp.

5 Constructive Homology and Commutative Algebra.

5.1 Presentation.

The homological framework was not available at Hilbert's time, but among his famous results in Commutative Algebra, typically the theorems about *syzygies*, many of them in fact have a *homological* nature. Henri Cartan and Sam Eilenberg [13] understood the various homological notions intensively used in Algebraic Topology can in fact be organized in a strictly exact and spectral sequences of Algebraic Topology as well.

We explain in this section how the point of view of *constructive* homological algebra gives new insights about some homological domains of commutative algebra. The following theme is particularly convenient. A classical theorem, the bicomplex spectral sequence, allows to prove the equivalence of both definitions of torsion groups:

$$H_*(\mathrm{Rsl}_{\mathfrak{R}}(M) \otimes_{\mathfrak{R}} N) =: \mathrm{Tor}^{\mathfrak{R}}_*(M, N) := H_*(M \otimes_{\mathfrak{R}} \mathrm{Rsl}_{\mathfrak{R}}(N)).$$

$\mathfrak{R}$ is a commutative unitary ground ring, M and N are two $\mathfrak{R}$-modules. The torsion groups of M and N are defined by taking for example an $\mathfrak{R}$-resolution $\mathrm{Rsl}(M)$ of M and computing the tensor product $\mathrm{Rsl}(M) \otimes_{\mathfrak{R}} N$; the last chain-complex in general is nomore exact, and its homology groups are the torsion groups. If you do the symmetric work with a resolution of N, the result is the same; the result is also independent of the chosen resolutions, so that these torsion groups express deep abstract relations between the modules M and N.

We intend to illustrate that a systematic *constructive* point of view in these homological notions produces new methods and also allows their users to have a more global understanding of the various studied properties. The last but not the least, most often the proofs are more elementary! We will so obtain the striking result: there is a perfect direct equivalence between the *effective* homology of $\mathrm{Ksz}(M)$, the Koszul complex of an $\mathfrak{R}$-module M, and a resolution $\mathrm{Rsl}_{\mathfrak{R}}(M)$ of the same module with respect to the ground ring $\mathfrak{R}$; in particular, the *minimal* effective homology corresponds to the *minimal* resolution.

5.2 Koszul complex.

UOStated 55 — *In this section about Commutative Algebra, the ground ring $\mathfrak{R}$ is $\mathfrak{R} = \mathfrak{k}[x_1, \ldots, x_m]_0$, the usual polynomial ring with m variables,* localized *at $0 \in \mathfrak{k}^m$. We denote by V the "abstract" vector space $V = \mathfrak{k}^m$ provided with the basis $(dx_1, \ldots, dx_m)$.*

The ground *field* $\mathfrak{k}$ is an arbitrary commutative field, in particular the case of a finite characteristic is covered without any extra work. An element of $\mathfrak{R}$ is a "quotient" P/Q of two polynomials, the second one being non-null at 0. It happens the denominators, because of the context, will not play any role, but the general correct framework is the case of $\mathfrak{R}$ a *regular local ring.* The basic reference about local rings is [60]. To make significantly more readable the exposition, we prefer to consider only the case of $\mathfrak{R} = \mathfrak{k}[x_1, \ldots, x_m]_0$.

Definition 56 — The *Koszul complex* $\mathrm{Ksz}(M)$ of the $\mathfrak{R}$-module M is a chain-complex of $\mathfrak{R}$-modules constructed as follows. The chain group in degree $n \geq 0$ is $\mathrm{Ksz}_n(M) = M \otimes_{\mathfrak{k}} \wedge^n V$ and the differential $d : \mathrm{Ksz}_n(M) \to \mathrm{Ksz}_{n-1}(M)$ is defined by the formula:

$$\begin{aligned} d(\alpha\, dx_{i_1} \ldots dx_{i_n}) &= \alpha x_{i_1}\, dx_{i_2} \ldots dx_{i_n} \\ &\quad -\alpha x_{i_2}\, dx_{i_1}.dx_{i_3} \ldots dx_{i_n} \\ &\quad + \cdots \\ &\quad +(-1)^{n-1} \alpha x_{i_n}\, dx_{i_1} \ldots dx_{i_{n-1}}. \end{aligned}$$

Observe we write simply $\alpha dx_2.dx_4.dx_5$ instead of $\alpha \otimes (dx_2 \wedge dx_4 \wedge dx_5)$ if $\alpha \in M$. The definition can be generalized to an arbitrary collection of elements $(\alpha_1, \ldots, \alpha_p)$ of $\mathfrak{R}$ instead of the "variables" $(x_1, \ldots, x_m)$; the differential of dx_i $(1 \leq i \leq p)$ is then α_i.

The usual sign game shows the Koszul complex actually is a chain-complex. Furthermore this will be also a *consequence* of a recursive construction given soon.

5.3 Geometrical interpretation.

The construction of a Koszul complex is a little strange, but becomes more natural if we give a geometrical interpretation, in fact historically at the origin of this notion [33]. This interpretation is never used later in this text.

In our environment, you must think of the ring $\mathfrak{R}$ as a topological group, used as a *structural group* to construct fibrations. The exterior algebra $\wedge V$ is also a *coalgebra* for the shuffle coproduct:

$$\Delta(v_1 \wedge \cdots \wedge v_k) = \sum (-1)^\sigma (v_{\sigma_1} \wedge \cdots \wedge v_{\sigma_\ell}) \otimes (v_{\sigma_{\ell+1}} \wedge \cdots \wedge v_{\sigma_k})$$

where the sum is taken with respect to all the shuffles $((\sigma_1 < \cdots < \sigma_\ell), (\sigma_{\ell+1} < \cdots < \sigma_k)$ for $0 \leq l \leq k$. The coalgebra structure of $\wedge V$ gives it a flavor of *topological space*, think of the Alexander-Whitney coproduct over the singular chain-complex of a topological space.

In the particular case $M = \mathfrak{R}$, the Koszul complex $\mathrm{Ksz}(\mathfrak{R})$ can be viewed as a principal fibration, the "base space" being $\wedge V$ and the "structural group" $\mathfrak{R}$. This is made more explicit in the notation $\mathrm{Ksz}(\mathfrak{R}) := \mathfrak{R} \otimes_t \wedge V$ to be understood as follows: the Koszul complex is a twisted (index t of $\otimes_t$) product of the base space $\wedge V$ by the structural group $\mathfrak{R}$, the twist t being defined by a *twisting cochain* $t \in H^1(\wedge V; \mathfrak{R})$; in the particular case of the Koszul complex, this twisting cochain is null outside the degree 1 component $\wedge^1 V$ of $\wedge V$ and $t(dx_i) := x_i$; see for example [40, § 30] for the general definition of the notion of twisting cochain. Such a twisting cochain is the translation in the algebraic framework of the *coordinate functions*, more precisely of the coordinate changes defining a fibre bundle [64, Section I.2].

Finally, if M is an arbitrary $\mathfrak{R}$-module, it can be understood as a topological space provided with an action $M \otimes_{\mathfrak{k}} \mathfrak{R} \rightarrow M$, which allows us to interpret the Koszul complex $\mathrm{Ksz}(M) = M \otimes_t \wedge V = M \otimes_{\mathfrak{R}} (\mathfrak{R} \otimes_t \wedge V)$ as the fibre bundle canonically associated with the principal bundle $\mathfrak{R} \otimes_t \wedge V$.

The chain-complex $\mathrm{Ksz}(\mathfrak{R})$ is acyclic, and we will see the *homotopy operator* proving this fact will play a quite essential role in our study. So that $\mathrm{Ksz}(\mathfrak{R})$ has the "homotopy type" of a point; in other words the fibration $\mathfrak{R} \otimes_t \wedge V$ is the *universal* $\mathfrak{R}$*-fibration*; in algebraic language $\mathrm{Ksz}(\mathfrak{R})$ is a *resolution* of the ground field $\mathfrak{k}$.

5.4 Tensor products of chain-complexes.

We give here a few general technical results about tensor products of chain-complexes and reductions. The ring $\mathfrak{R}$ *in this subsection* is again an arbitrary commutative unitary ring.

Definition 57 (Koszul convention) — Let C_* and D_* be two graded modules $C_* = \oplus_n C_n$ and $D_* = \oplus_n D_n$. A natural graduation is induced over $T_* = C_* \otimes D_* = \oplus_n(\oplus_{p+q=n} C_p \otimes D_q)$. If $f : C_* \rightarrow C'_{*+k}$ and $g : D_* \rightarrow D'_{*+\ell}$ are graded morphisms of respective degrees k and ℓ, then the tensor product $f \otimes g : (C \otimes D)_* \rightarrow (C' \otimes D')_{*+k+l}$ is defined by $(f \otimes g)(a \otimes b) := (-1)^{\ell|a|} f(a) \otimes g(b)$ if a is homogeneous of degree $|a|$.

We think the necessary permutation of g (degree ℓ) and a (degree $|a|$) generates a signature $(-1)^{\ell|a|}$.

Definition 58 — Let (C_*, d) and (C'_*, d') be two chain-complexes of $\mathfrak{R}$-modules. The tensor product $(C_*, d) \otimes (C'_*, d')$ is a chain-complex defined as the module $(C_* \otimes C'_*)$ provided with the differential $d_{C_* \otimes C'_*} := d \otimes \mathrm{id}_{C'_*} + \mathrm{id}_{C_*} \otimes d'$ where the Koszul convention must be applied. The identity being of degree 0 and a differential of degree -1, this implies $d(a \otimes b) = da \otimes b + (-1)^{|a|} a \otimes db$.

The tensor product operator is an important functor and we must be able to define the tensor product of two reductions. It is better to start with the composition of reductions.

Proposition 59 — *Let $\rho = (f, g, h) : C_* \Rrightarrow C'_*$ and $\rho' = (f', g', h') : C'_* \Rrightarrow C''_*$ be two reductions. These reductions can be* composed*, producing the reduction $\rho'' = (f'', g'', h'') : C_* \Rrightarrow C''_*$ with:*

$$\begin{array}{rcl} f'' & = & f'f; \\ g'' & = & gg'; \\ h'' & = & h + gh'f. \end{array}$$

Proof. Exercise. ∎

Proposition 60 — *Let $\rho = (f, g, h) : C_* \Rrightarrow D_*$ and $\rho' = (f', g', h') : C'_* \Rrightarrow D'_*$ be two reductions. Then a tensor product:*

$$\rho'' = (f'', g'', h'') : C_* \otimes C'_* \Rrightarrow D_* \otimes D'_*$$

can be defined, with:

$$\begin{array}{rcl} f'' & = & f \otimes f'; \\ g'' & = & g \otimes g'; \\ h'' & = & h \otimes id_{C'_*} + gf \otimes h' \end{array}$$

Proof. Compose the reductions

$$\begin{array}{rcl} \rho \otimes \mathrm{id}_{C'_*} & : & C_* \otimes C'_* \Rrightarrow D_* \otimes C'_* \\ \mathrm{id}_{D_*} \otimes \rho' & : & D_* \otimes C'_* \Rrightarrow D_* \otimes D'_*. \end{array}$$

∎

Note the lack of symmetry in the result; you could replace the intermediate complex by $C_* \otimes D'_*$ and $h'' = h \otimes \mathrm{id}_{C'_*} + gf \otimes h'$ by $h'' = \mathrm{id}_{C_*} \otimes h' + h \otimes g'f'$.

5.5 Cones of chain-complexes.

The *cone constructor* is important in homological algebra, and we study here the most elementary properties. We will meet the first application of the BPL.

Definition 61 — Let C_* and D_* be two chain-complexes and $\phi : C_* \leftarrow D_*$ be a chain-complex morphism. Then the cone of ϕ denoted by $\mathrm{Cone}(\phi)$ is the chain-complex $\mathrm{Cone}(\phi) = A_*$ defined as follows. First $A_n := C_n \oplus D_{n-1}$; then the boundary operator is given by the matrix:

$$d_{A_*} := \begin{bmatrix} d_{C_*} & \phi \\ 0 & -d_{D_*} \end{bmatrix}$$

We prefer to turn to the left the arrow from D_* to C_*, because a cone is in fact a particular case of a bicomplex and experience shows it is convenient to keep one's organisation as homogeneous as possible. The diagram clearly explaining the nature of a cone is the following.

$$\begin{tikzcd}
\cdots & D_{n-2} \arrow[l, "-d_D"'] \arrow[dl, "\phi"] \arrow[d, dash, dotted, "\oplus"] & D_{n-1} \arrow[l, "-d_D"'] \arrow[dl, "\phi"] \arrow[d, dash, dotted, "\oplus"] & D_n \arrow[l, "-d_D"'] \arrow[dl, "\phi"] \arrow[d, dash, dotted, "\oplus"] & \cdots \arrow[l, "-d_D"'] \arrow[dl, "\phi"] \\
\cdots & C_{n-1} \arrow[l, "d_C"] & C_n \arrow[l, "d_C"] & C_{n+1} \arrow[l, "d_C"] & \cdots \arrow[l, "d_C"]
\end{tikzcd}$$

You see the morphism ϕ contributes to the differential of the cone. If you do not change the sign of d_{D_*} in the cone the rule $d \circ d = 0$ would not be satisfied. With our sign choice:

$$\begin{bmatrix} d & \phi \\ 0 & -d \end{bmatrix} \begin{bmatrix} d & \phi \\ 0 & -d \end{bmatrix} = \begin{bmatrix} d^2 & d\phi - \phi d \\ 0 & d^2 \end{bmatrix} = 0.$$

for the initial differentials satisfy $d^2 = 0$ and ϕ is a chain-complex morphism satisfying $d_C\phi = \phi d_D$. In fact the Koszul convention has been applied: the suspension operator σ which increases the degree by 1 is implicitly applied to the elements of D_* and this suspension operator has degree $+1$. So that Koszul teaches us that $d(\sigma c) = -\sigma(dc)$ is the good choice: the morphisms σ (suspension, degree $+1$) and d (differential, degree -1) have been permuted.

Studying carefully the next simple application of the BPL (basic perturbation lemma) gives an excellent understanding of this wonderful Theorem strangely called "lemma". This application is not difficult; all the applications have the same style and this one is the simplest one. Consider this particular case as the ideal didactic situation to *learn* how to use the BPL; the other applications are not more difficult, even in more or less terrible environments.

Theorem 62 (Cone Reduction Theorem) — *Let $\rho = (f, g, h) : C_* \Rrightarrow D_*$ and $\rho' = (f', g', h') : C'_* \Rrightarrow D'_*$ be two reductions and $\phi : C_* \leftarrow C'_*$ a chain-complex morphism. Then these data define a canonical reduction:*

$$\rho'' = (f'', g'', h'') : \mathit{Cone}(\phi) \Rrightarrow \mathit{Cone}(f\phi g').$$

Proof. This would be trivial if $\phi = 0$: in such a case, we have also $f\phi g' = 0$ and the cones are simple direct sums (with a suspension applied over C'_* and D'_*) and defining a direct sum of reductions is trivial. Now look carefully at this diagram:

$$\begin{tikzcd}
C_* \arrow[loop left, "h"] \arrow[d, shift right, "f"'] & C'_* \arrow[l, dotted, "0"'] \arrow[loop right, "h'"] \arrow[d, shift right, "f'"'] \\
D_* \arrow[u, shift right, "g"'] & D'_* \arrow[l, dotted, "0"'] \arrow[u, shift right, "g'"']
\end{tikzcd}$$

The rectangular boxes intend to visualize the cone constructions, simple direct sums when the chain-complex morphisms are null. The suspensions applied to the right-hand chain-complexes are not shown. Each chain-complex of these (trivial) cones is a direct sum, so that the morphisms of our initial reduction are represented by 2×2 matrices:

$$\underset{d_{\text{top}}}{\begin{bmatrix} d_C & 0 \\ 0 & -d'_{C'} \end{bmatrix}} \quad \underset{d_{\text{bottom}}}{\begin{bmatrix} d_D & 0 \\ 0 & -d'_{D'} \end{bmatrix}} \quad \underset{f \oplus f'}{\begin{bmatrix} f & 0 \\ 0 & f' \end{bmatrix}} \quad \underset{g \oplus g'}{\begin{bmatrix} g & 0 \\ 0 & g' \end{bmatrix}} \quad \underset{h \oplus -h'}{\begin{bmatrix} h & 0 \\ 0 & -h' \end{bmatrix}}$$

A homotopy operator has degree $+1$ and the Koszul convention must also be applied between suspension and homotopy. Now if we install the right morphism ϕ on the top cone, the reduction is nomore valid, the top differential is modified and there is no reason the pairs $f \oplus f'$ and $g \oplus g'$ are compatible with the new top differential. It is exactly in such a situation the BPL is to be used. In this case the *perturbation* to be applied to d_{top} is:

$$\begin{bmatrix} d_C & 0 \\ 0 & -d'_{C'} \end{bmatrix} + \begin{bmatrix} 0 & \phi \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} d_C & \phi \\ 0 & -d'_{C'} \end{bmatrix}$$

This is frequent in applications of the BPL, the perturbations are extra arrows installed in the diagram after the starting situation, here only one arrow ϕ. The BPL can be used only if the nilpotency condition is satisfied. The composition $h\widehat{d}$ of Theorem 50 is here:

$$\begin{bmatrix} h & 0 \\ 0 & -h'_{C'} \end{bmatrix} \circ \begin{bmatrix} 0 & \phi \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & h\phi \\ 0 & 0 \end{bmatrix}$$

which is clearly nilpotent. Instead of formal computations, verifying the nilpotency condition is most often the following game: follow a perturbation arrow, then a homotopy arrow, then a perturbation arrow, and so on. You must show this treasure hunt, in general several possible choices at each step, terminates after a finite number of steps, whatever your choices are. Here the longest path is $h\phi h'$ and it is not possible to extend this path of length 3, the nilpotency condition is therefore satisfied.

We remind you of the magic Shih's formulas in the general framework of Theorem 50, in particular with the notations of Theorem 50:

$$\phi = \sum_{i=0}^{\infty} (-1)^i (h\widehat{\delta})^i; \qquad \psi = \sum_{i=0}^{\infty} (-1)^i (\widehat{\delta} h)^i.$$

$$\delta = f\widehat{\delta}\phi g = f\psi\widehat{\delta}g; \quad f' = f\psi; \quad g' = \phi g; \quad h' = \phi h = h\psi.$$

Applying these formulas to our particular situation gives:

$$\phi_{\text{Shih}} = \begin{bmatrix} 1 & -h\phi \\ 0 & 1 \end{bmatrix}; \qquad \psi_{\text{Shih}} = \begin{bmatrix} 1 & -\phi h' \\ 0 & 1 \end{bmatrix};$$

and then, with our current notations, except δ being the perturbation to apply to the bottom cone:

$$\delta = \begin{bmatrix} 0 & f\phi g' \\ 0 & 0 \end{bmatrix}; f'' = \begin{bmatrix} f & -f\phi h' \\ 0 & f' \end{bmatrix}; g'' = \begin{bmatrix} g & -h\phi g' \\ 0 & g' \end{bmatrix}; h'' = \begin{bmatrix} h & h\phi h' \\ 0 & -h' \end{bmatrix};$$

In other words we have successfully constructed the right new reduction between $\text{Cone}(\phi)$ and $\text{Cone}(f\phi g')$:

$$\begin{tikzcd}
C_* \arrow[loop left, "h"] \arrow[d, shift right, "f"'] & C'_* \arrow[l, "\phi"'] \arrow[l, bend right, "h\phi h'"'] \arrow[loop right, "-h'"] \arrow[d, shift right, "f'"'] \arrow[dl, "-f\phi h'"] \\
D_* \arrow[u, shift right, "g"'] & D'_* \arrow[l, "f\phi g'"] \arrow[u, shift right, "g'"'] \arrow[ul, "-h\phi g'"]
\end{tikzcd}$$

■

Of course, most often this theorem is proved *without* using the BPL, but experience shows it is not so easy to guess the right compositions and the right signs. Once the BPL is understood, it is easier to use it to prove the cone theorem.

5.6 Resolutions.

In this section which has a general scope, the ring $\mathfrak{R}$ is an arbitrary unitary commutative ring.

Definition 63 — Let M be an $\mathfrak{R}$-module. A *free* $\mathfrak{R}$*-resolution* of M, in short a *resolution* of M, is a chain-complex $\text{Rsl}(M)$ null in negative degrees, made of *free* $\mathfrak{R}$-modules, every differential is an $\mathfrak{R}$-morphism, every homology group $H_n(\text{Rsl}(M))$ is null except $H_0(\text{Rsl}(M))$: an $\mathfrak{R}$-isomorphism $\varepsilon : H_0(\text{Rsl}(M)) \cong M$ is *given.*

Note the isomorphism is a component of the data defining the resolution; strictly speaking the resolution is the *pair* $(\text{Rsl}(M), \varepsilon)$. You can also consider the isomorphism ε as coming from a morphism called *augmentation* $\overline{\varepsilon} : \text{Rsl}_0(M) \to M$. If you "add" $\text{Rsl}_{-1}(M) := M$ and this augmentation, you obtain the exact sequence:

$$0 \leftarrow M \overset{\overline{\varepsilon}}{\leftarrow} \text{Rsl}_0(M) \leftarrow \text{Rsl}_1(M) \leftarrow \cdots$$

but the good point of view is not to include M which must be isomorphic to the H_0-group of the resolution, with a *given* isomorphism. The functional notation $\mathrm{Rsl}(M)$ is justified by the fact such a resolution is unique up to homotopy, a point not very important here.

Another detail about notations in this context must be given. Sometimes the module M is better considered as a chain-complex concentrated in dimension 0, with null differentials, in particular when we will soon consider the notion of *effective* resolution; to emphasize this point of view, we sometimes use the $*$-notation $M_* := [\cdots \stackrel{0}{\leftarrow} 0 \stackrel{0}{\leftarrow} M \stackrel{0}{\leftarrow} 0 \stackrel{0}{\leftarrow} \cdots]$. Both points of view have their own interest and it is not always possible to keep a constant notation.

We want to make *effective* (or constructive) the definition of a resolution. We want to make explicit a contracting homotopy *proving* the very nature of our resolution. In general there is no hope these homotopy operators are $\mathfrak{R}$-morphisms. To keep some linear behaviour, *we assume now* our ground ring $\mathfrak{R}$ is a $\mathfrak{k}$-algebra with respect to a commutative field $\mathfrak{k}$. It is the case for the usual rings of commutative algebra, for example for the ring $\mathfrak{k}[x_1, \ldots, x_m]_0$.

Definition 64 — Let M be an $\mathfrak{R}$-module. An *effective* resolution $\mathrm{Rsl}(M)$ is a resolution with a $(\mathfrak{R}, \mathfrak{k}, \mathfrak{k})$*-reduction* $\rho = (f, g, h) : \mathrm{Rsl}(M) \Lleftarrow\!\!\!\Rrightarrow M_*$ where the small chain-complex M_* is made from M concentrated in degree 0.

The prefix $(\mathfrak{R}, \mathfrak{k}, \mathfrak{k})$ for our reduction means we require f is an $\mathfrak{R}$-morphism, but g and h in general are only $\mathfrak{k}$-morphisms.

Example. Let us consider $\mathfrak{R} = \mathfrak{k}[x]_0$ (one variable). The evaluation $\mathrm{ev}_0(P) = P(0)$ gives a structure of $\mathfrak{R}$-module to $\mathfrak{k}$: $(P, k) \mapsto P(0)k$. What about a resolution of $\mathfrak{k}$?

The Koszul complex $\mathrm{Ksz}(\mathfrak{R})$ is in this case very simple; it is a chain-complex concentrated in degrees 0 and 1:

$$0 \leftarrow \mathfrak{R} \stackrel{d_1}{\leftarrow} \mathfrak{R}.dx \leftarrow 0$$

with $d_1(P.dx) = Px$ ($P \times x$, not P(x)). This d_1 is injective, and $H_1 = 0$. The image is the maximal ideal $\mathfrak{m}$ and therefore $H_0 = \mathfrak{R}/\mathfrak{m} = \mathfrak{k}$. The Koszul complex is a *resolution of* $\mathfrak{k}$.

But we are not happy with this result, we prefer *effective* resolutions. Can this resolution be made effective? It is not hard. The projection $f : \mathrm{Ksz}(\mathfrak{R}) \to \mathfrak{k}_*$ is given by the composition $\mathrm{Ksz}_0(\mathfrak{R}) \to \mathrm{Ksz}_0(\mathfrak{R})/d_1(\mathrm{Ksz}_1(\mathfrak{R})) = \mathfrak{R}/\mathfrak{m} = \mathfrak{k}$. The inclusion $g : \mathfrak{k} \to \mathrm{Ksz}_0(\mathfrak{R}) = \mathfrak{R}$ is the canonical inclusion $\mathfrak{k} \hookrightarrow \mathfrak{R}$ *which is not* an $\mathfrak{R}$-morphism. Finally the homotopy operator must be defined in degree 0, the most natural choice being $h_0(P) = (P - P(0))/x$, which is not an $\mathfrak{R}$-morphism either: $h_0(1) = 0$ and $h_0(x) = 1 \neq xh_0(1) = 0$. But h_0 is $\mathfrak{k}$-linear.

In a sense we want to extend the elementary study of this example to the general case. We want to prove that, if $\mathfrak{R} = \mathfrak{k}[x_1, \ldots, x_m]_0$, then $\mathrm{Ksz}(\mathfrak{R})$ is an *effective free* $\mathfrak{R}$*-resolution* of the ground field $\mathfrak{k}$. The proof is inductive, easy if the polynomial ring is not localized [36, Proposition VII.2.1], a little harder but also a little more interesting in the localized case[22]. We must precisely connect our various rings for different numbers of variables.

Notation 65 — *The number m of variables we are interested in is fixed. If $0 \leq q \leq m$, we denote by I_q the ideal of $\mathfrak{R} = \mathfrak{k}[x_1, \ldots, x_m]_0$ generated by the variables $x_{q+1}, \ldots, x_m$: $I_q = <x_{q+1}, \ldots, x_m>$. The quotient ring $\mathfrak{R}/I_q$ is denoted by $\mathfrak{R}_q$. We denote by V_q the $\mathfrak{k}$-vector space of dimension $m - q$ generated by the distinguised basis $(dx_{q+1}, \ldots, dx_m)$.*

The ring $\mathfrak{R}_q$ is the same as $\mathfrak{R}$ except any occurence of a variable x_r with $r > q$ is cancelled. So that $\mathfrak{R}_q$ is the analogous local ring but with q variables only. In particular $\mathfrak{R} = \mathfrak{R}_m$ and $\mathfrak{k} = \mathfrak{R}_0$. If $q \leq r$, canonical morphisms $f_{q,r} : \mathfrak{R}_r \to \mathfrak{R}_q$ and $g_{q,r} : \mathfrak{R}_q \to \mathfrak{R}_r$ are defined. The first one is a projection, it is also an evaluation process consisting in replacing the variables $x_{q+1}, \ldots, x_r$ by 0; it is an $\mathfrak{R}_i$-morphism for every i, in particular for $i = m$. The second one is a canonical inclusion, it is an $\mathfrak{R}_i$-morphism only for $i \leq q$.

Definition 66 — The definition of the Koszul complex is extended as follows. We denote by $\mathrm{Ksz}^q(M)$ the sub-chain-complex $\mathrm{Ksz}^q_k(M) = M \otimes_{\mathfrak{k}} \wedge^k V_q$ of $\mathrm{Ksz}(M)$.

The only difference between $\mathrm{Ksz}^q(M)$ and $\mathrm{Ksz}(M)$ is that in the first case a dx_i with $i \leq q$ is excluded.

Theorem 67 — $\mathrm{Ksz}(\mathfrak{R})$ *is an effective free $\mathfrak{R}$-resolution of the $\mathfrak{R}$-module $\mathfrak{k}$.*

It is the particular case $q = 0$ of the next theorem to be proved by decreasing induction.

Theorem 68 — $\mathrm{Ksz}^q(\mathfrak{R})$ *is an effective free $\mathfrak{R}$-resolution of the $\mathfrak{R}$-module $\mathfrak{R}_q$.*

Note strictly speaking such a statement is improper. When we claim some object is *effective*, we mean some collection of algorithms, more or less difficult to be constructed, will allow us to justify the qualifier.

Proof. The theorem is obvious for $q = m$: the chain-complex $0 \leftarrow \mathfrak{R} \leftarrow 0$ concentrated in degree 0 is a resolution of $\mathfrak{R}$.

Let us assume the theorem is proved for q and let us prove it for $q - 1$. A reduction $\rho_q = (f_q, g_q, h_q) : \mathrm{Ksz}^q(\mathfrak{R}) \Rrightarrow \mathfrak{R}_q$ is available.

Our simple example above is easily adapted to prove:

Lemma 69 — *The chain-complex*

$$0 \leftarrow \mathfrak{R}_q \xleftarrow{\times x_q} \mathfrak{R}_q \leftarrow 0$$

is an effective free resolution of $\mathfrak{R}_{q-1}$.

[22] We could also use the *flatness* property of $\mathfrak{R}$ as $\overline{\mathfrak{R}}$-module, but an *effective* flatness is required; see [45, III.5] for the right definition.

It is a sophisticated and precise way to express the map $\times x_q$ is injective and its cokernel is $\mathfrak{R}_{q-1}$. The relevant reduction is made of the projection $f_{q-1,q}$ which is an $\mathfrak{R}$-morphism, the injection $g_{q-1,q}$ which is an $\mathfrak{R}_{q-1}$-morphism only, and the homotopy operator $h_0(\alpha) = (\alpha - \alpha(x_q = 0)/x_q)$ which is an $\mathfrak{R}_{q-1}$-morphism. ■

Proof of Theorem continued. Thanks to the reduction ρ_q, the object $\mathrm{Ksz}^q(\mathfrak{R})$ is "above" $\mathfrak{R}_q$. The morphism $\times x_q$ is trivially lifted into a chain-complex morphism: $\times x_q : \mathrm{Ksz}^q(\mathfrak{R}) \leftarrow \mathrm{Ksz}^q(\mathfrak{R})$; the source and the target of this morphism are reduced through ρ_q over $\mathfrak{R}_q$ and we can apply the Cone Reduction Theorem 62. Combining with the other reduction already available, we obtain:

$$\mathrm{Cone}(\mathrm{Ksz}^q(\mathfrak{R}) \xleftarrow{\times x_q} \mathrm{Ksz}^q(\mathfrak{R})) \Rrightarrow \mathrm{Cone}(\mathfrak{R}_{q,*} \xleftarrow{\times x_q} \mathfrak{R}_{q,*}) \Rrightarrow \mathfrak{R}_{q-1}$$

where the $\mathfrak{R}_{q,*}$ terms are understood as chain-complexes concentrated in degree 0. Composing both reductions (Proposition 59) gives the result if we can identify the first cone with $\mathrm{Ksz}^{q-1}(\mathfrak{R})$. This cone is made of two copies of $\mathrm{Ksz}^q(\mathfrak{R})$; to distinguish them, let us recall the right hand one $dx_q.\mathrm{Ksz}^q(\mathfrak{R})$, that is, for every term of this $\mathrm{Ksz}^q(\mathfrak{R})$, let us put a symbol dx_q between the coefficient in $\mathfrak{R}$ and the exterior part in $\wedge V_q$. This increases the Koszul degree in the chain-complex by $+1$, but by chance the right hand term in a cone is suspended. When you compute the differential of $\alpha\, dx_q \ldots$ in a Koszul complex, the contribution of dx_q corresponds here to our $\times x_q$ morphism, the other terms come from the differential of $\mathrm{Ksz}^q(\mathfrak{R})$. In fact with another sign, but the sign of the differential in the right hand component of a cone is also changed. Conclusion: there is a natural canonical isomorphism of chain-complex $\mathrm{Cone}(\mathrm{Ksz}^q(\mathfrak{R}) \xleftarrow{\times x_q} \mathrm{Ksz}^q(\mathfrak{R})) \cong \mathrm{Ksz}^{q-1}(\mathfrak{R})$. ■

A novice can be troubled by the following observation: more $\mathfrak{R}_q$ is small, more $\mathrm{Ksz}^q(\mathfrak{R})$ is big? The point is that if $\mathfrak{R}_q$ is smaller, then the "difference" between the ground ring $\mathfrak{R}$ and $\mathfrak{R}_q$ is bigger, so that the resolution is logically more complicated. The proof start from $\mathfrak{R}$ and goes up to $\mathfrak{k}$ through the various $\mathfrak{R}_q$.

From a computational point of view, it is important to make explicit the homotopy component h of the reduction $\mathrm{Ksz}(\mathfrak{R}) \Rrightarrow \mathfrak{k}$. Using the detailed formula given when proving the Cone Reduction Theorem 62, it is easy to prove our homotopy operator is given by the formula:

$$h(\alpha.\lambda) = \sum_{q=1}^{m} ((\alpha(x_1, \ldots, x_q, 0, \ldots, 0) - \alpha(x_1, \ldots, x_{q-1}, 0, \ldots, 0)/x_q)\ dx_q.\lambda$$

if $\alpha \in \mathfrak{R}$ and $\lambda \in \wedge V$ with the common interpretations inside the exterior algebra $\wedge V$: if ever dx_q is present in λ, then $dx_q.\lambda = 0$; and if dx_q is at a wrong place, putting it at the right place can need a sign change. It is amusing to study the particular case where α is a monomial $\alpha = x_{i_1}^{j_1} \cdots x_{i_k}^{j_k}$ with $i_1 < \cdots < i_k$ and $j_k > 0$. If dx_{i_k} is present in λ, the result is 0; otherwise you replace j_k by $j_k - 1$ in the monomial and you insert a dx_{i_k} in λ at the right place with the right sign. In concrete programming, this can be run very efficiently. We will see this algorithm is by far the most used in the resulting programs. Because of the proverb: the difference between effective homology and ordinary homology consists in using the *explicit* homotopy operators.

5.7 Koszul complex with effective homology

The reduction constructed in the previous section can be understood as describing the *effective homology* of our Koszul complex.

Theorem 70 — *If $\mathfrak{R}$ is the ring $\mathfrak{R} = \mathfrak{k}[x_1, \ldots, x_m]_0$, the Koszul complex* $\mathrm{Ksz}(\mathfrak{R})$ *"is" an object with effective homology.*

Proof. As usual, strictly speaking, the statement is improper: the statement part "is an object ..." is a shorthand; in fact it is claimed some process allows us to complete the object under study, the Koszul complex, as a quadruple satisfying the required rules of Definition 53. This quadruple is $(\mathrm{Ksz}(\mathfrak{R}), \mathrm{Ksz}(\mathfrak{R}), \mathfrak{k}_*, \rho)$ where ρ is the reduction of the previous section considered as the equivalence:

$$\mathrm{Ksz}(\mathfrak{R}) \overset{=}{\lll} \mathrm{Ksz}(\mathfrak{R}) \overset{\rho}{\ggg} \mathfrak{k}_*.$$

■

5.8 Torsion groups.

Definition 71 — Let $\mathfrak{R} = \mathfrak{k}[x_1, \ldots, x_m]_0$ be our traditional ring and let M and N be two $\mathfrak{R}$-modules. The torsion groups $\mathrm{Tor}_i^{\mathfrak{R}}(M, N)$ are defined as follows. Let $\mathrm{Rsl}(M)$ and $\mathrm{Rsl}(N)$ be two (free) $\mathfrak{R}$-resolutions of M and N. Then:

$$H_*(\mathrm{Rsl}_{\mathfrak{R}}(M) \otimes_{\mathfrak{R}} N) =: \mathrm{Tor}_*^{\mathfrak{R}}(M, N) := H_*(M \otimes_{\mathfrak{R}} \mathrm{Rsl}_{\mathfrak{R}}(N)).$$

It is not obvious the definition is *coherent*, that is, the result does not depend on the choice of using a resolution of M or N, and does not depend either on the choice of the resolution itself. The usual argument uses the bicomplex spectral sequence, and it is a good opportunity to introduce the effective version of this spectral sequence.

Definition 72 — A first quadrant *bicomplex* is a diagram of modules:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
\cdots & C_{p-1,q} \arrow[l] \arrow[d, "d''"'] & C_{p,q} \arrow[l, "d'"'] \arrow[d, "d''"'] & \cdots \arrow[l] \\
\cdots & C_{p-1,q-1} \arrow[l] \arrow[d] & C_{p,q-1} \arrow[l, "d'"'] \arrow[d] & \cdots \arrow[l] \\
 & {} & {} &
\end{tikzcd}$$

with $C_{p,q} = 0$ if p or $q < 0$. Furthermore every horizontal is a chain-complex ($d'd' = 0$), every vertical is a chain-complex ($d''d'' = 0$), and every square is *anti-commutative*: $d'd'' + d''d' = 0$. The totalization of this bicomplex is a *simple* chain-complex (T_n, d_n) where $T_n = \oplus_{p+q=n} C_{p,q}$ and the differential dc of a chain $c \in C_{p,q} \subset T_{p+q}$ is $dc = d'c \oplus d''c \in C_{p-1,q} \oplus C_{p,q-1} \subset T_{p+q-1}$.

The relations required for d' and d'' are exactly the necessary relations which do make the totalization a chain-complex. The bicomplex spectral sequence gives a relation between the homology of every column (for example) and the homology of the totalization. Other similar definitions can be given for other quadrants or for the whole (p, q)-plane.

Theorem 73 (Bicomplex Spectral Sequence)— *If $(C_{p,q}, d'_{p,q}, d''_{p,q})$ is a first quadrant bicomplex, a spectral sequence $(E^r_{p,q}, d^r_{p,q})$ can be defined with $E^0_{p,q} = C_{p,q}$, $d^0_{p,q} = d''_{p,q}$, and $E^1_{p,q} = H''_{p,q}$ is the "vertical" homology group of the p-column at index q. Furthermore this spectral sequence converges to the homology of the totalization:*

$$E^r_{p,q} \Rightarrow H_{p+q}(T_*).$$

Of course you can exchange the role of rows and columns and obtain *another* spectral sequence where $E^1_{p,q} = H'_{p,q}$ is this time the homology of the q-row at the index p, converging exactly toward the same homology groups $H_*(T_*)$. For the proof, see for example [36, Section XI.6] where the $E^2_{p,q}$ are also computed, quite elementary. Which is a little harder is $E^r_{p,q}$ for $r > 2$, a problem which gets *constructive* answers with our constructive methods.

Definition 74 — A first quadrant multicomplex $(C_{p,q}, d^r_{p,q})$ *is a collection of (p, q)-modules as for a bicomplex, but a large collection of (r, p, q)-arrows defined for $r \geq 0$ and every (p, q); the r-index describes the horizontal shift: $d^r_{p,q} : C_{p,q} \to C_{p-r,q+r-1}$. Furthermore, the natural totalization of these data must be a chain-complex.*

There are striking analogies with spectral sequences but also important differences. The most important difference is the following: the parameter r is nomore a "time" parameter, that is, all the arrows $d^r_{p,q}$ *coexist at the same time*, and anyway no time in this definition! The modules $C_{p,q}$ do not depend on r like the $E^r_{p,q}$ of a spectral sequence.

To define the totalization of a multicomplex, the process is as follows. As for a bicomplex, $T_n = \oplus_{p+q=n} C_{p,q}$. A component of the differential starts from every $C_{p,q}$ and goes to every $C_{p-r,q+r-1}$ for $r \geq 0$. A formula can be written:

$$(d : T_n \to T_{n-1}) = \oplus_{p+q=n}(\oplus_{r\geq 0} d^r_{p,q})$$

where the first $\oplus$ takes account of the expression of the *source* as a direct sum and the second $\oplus$ is analogous for the *target*. The relevant explicative diagram maybe is this one.

$$
\begin{tikzcd}
C_{p-3,q+2} \arrow[d, "d^0"] & & & \\
C_{p-3,q+1} & C_{p-2,q+1} \arrow[l, "d^1"'] \arrow[d, "d^0"] & & \\
& C_{p-2,q} & C_{p-1,q} \arrow[l, "d^1"'] \arrow[d, "d^0"] \arrow[ull, "d^2"'] & C_{p,q} \arrow[l, "d^1"'] \arrow[d, "d^0"] \arrow[ull, "d^2"'] \arrow[uulll, "d^3"'] \\
& & C_{p-1,q-1} & C_{p,q-1} \arrow[l, "d^1"'] \arrow[d, "d0"] \arrow[ull, "d^2"'] \arrow[uulll, "d^3"'] \\
& & & C_{p,q-2}
\end{tikzcd}
$$

This diagram intends to study what happens when starting from $C_{p,q}$. Observe there is a unique way in our treasure hunt diagram starting from $C_{p,q}$ and arriving at $C_{p,q-2}$. This implies necessarily $d^0 d^0 = 0$ and therefore *the columns are chain-complexes*. There are two ways reaching $C_{p-1,q-1}$ and we find again the anticommutativity property of the squares. But now there are three ways leading to $C_{p-2,q}$ and the totalisation will actually be a differential only if $d^0 d^2 + d^1 d^1 + d^2 d^0 = 0$. And so on. In general $\sum_{i=0}^{k} d^{k-i}_{p-i,q+i-1} d^i_{p,q} = 0$ is required for every (p, q) and every $k \geq 0$.

Theorem 75 (Bicomplex Reduction Theorem) — *Let $(C_{p,q}, d'_{p,q}, d''_{p,q})$ be a bicomplex and T_* be its totalization. Let $\rho_p = (f_p, g_p, h_p) : C_{p,*} \Rrightarrow D_{p,*}$ be a reduction of the (p)-column* given *for every p. Then a multi-complex $(D_{p,q}, d^r_{p,q})$ can be defined with the following property: let U_* be the totalization of this multicomplex; the reductions ρ_p defines a "total reduction" $\rho : T_* \Rrightarrow U_*$.*

Note the Cone Reduction Theorem 62 is in fact a particular case: if the bicomplex is null for $p \geq 2$, which remains is simply the cone of the columns 0 and 1; more precisely, in column 1, you must consider the chain-complex with an opposite (vertical) differential; the morphism defining the cone is given by the $d'_{1,q}$ arrows.

We explain after the proof in which circumstance this theorem is mainly used.

Proof. The proof is also a simple extension of the proof for the Cone Reduction Theorem. You consider firstly the same bicomplex but with all the horizontal differentials cancelled: $d'_{*,*} = 0$. Then the *different* totalization, let us call it $(T'_*, d_{T'})$, is nothing but the direct sum of the columns. The given reductions of the columns produce a reduction $\rho' = \oplus_p \rho_p : (T'_*, d_{T'}) \Rrightarrow (U'_*, d_{U'})$ with $U'_* = \oplus_p D_{p,*}$. This being observed, let us reinstall now the right horizontal arrows over T'_* to obtain again T_*; this can be viewed as a global *perturbation* of the differential $d_{T'}$ to obtain the differential d_T. Can we apply the BPL?

We must verify the nilpotency hypothesis. We must prove the composition $h\widehat{\delta} =$ homotopy-perturbation is pointwise nilpotent. Let us start from $C_{p,q}$. The perturbation $\widehat{\delta} = d'_{p,q}$ in this case leads to $C_{p-1,q}$, the homotopy operator to $C_{p-1,q+1}$.

If we repeat, we go to $C_{p-2,q+2}$, and so on, and after p steps, we reach $C_{0,p+q}$, but here the perturbation is null and the nilpotency hypothesis is satisfied. The role of the *first quadrant* property then is clear: the snake path must lead to some 0 module, whatever the starting point is.

$$\begin{tikzcd}
{[\text{nilpotency}]} & C_{0,p+q} \arrow[l, "\widehat{\delta}=0"'] & & \\
& C_{0,p+q-1} \arrow[u, "h"'] & C_{1,p+q-1} \arrow[l, "\widehat{\delta}"'] & \\
& & C_{1,p+q-2} \arrow[u, "h"'] & C_{2,p+q-2} \arrow[l, "\widehat{\delta}"'] \\
& & & {} \arrow[u, dotted, "h"']
\end{tikzcd}$$

Examining the nilpotency hypothesis gives also a good idea about the nature of the BPL series:

$$\phi = \sum_{i=0}^{\infty} (-1)^i (h\widehat{\delta})^i; \qquad\qquad \psi = \sum_{i=0}^{\infty} (-1)^i (\widehat{\delta}h)^i.$$

They are the sums of all the terms obtained following our snake paths, starting in the horizontal direction for ϕ, in the vertical direction for ψ. The next diagram shows the terms corresponding to $i = 2$, called ϕ_2, in dashed arrows, and ψ_2, solid arrows.

$$\begin{tikzcd}
C_{p-2,q+2} & C_{p-1,q+2} \arrow[l, "\widehat{\delta}"'] & \\
C_{p-2,q+1} \arrow[u, dashed, "h"] & C_{p-1,q+1} \arrow[l, dashed, "\widehat{\delta}"'] \arrow[u, "h"] & C_{p,q+1} \arrow[l, "\widehat{\delta}"'] \\
& C_{p-1,q} \arrow[u, dashed, "h"'] & C_{p,q} \arrow[l, dashed, "\widehat{\delta}"] \arrow[u, "h"'] \arrow[uull, bend right, "\psi_2"'] \arrow[uull, dashed, bend left, "\phi_2"]
\end{tikzcd}$$

Then these series ϕ and ψ have to be combined with the original f, g, h, $d_{T'}$ and $d_{U'}$ to produce the looked-for reduction between T_*, given, and U_*, to be constructed. This is the role of BPL. In particular, taking account of the formula $\delta = f\widehat{\delta}\phi g$ for the resulting perturbation on the small complex U'_* transforming it into the perturbed small one U_*, you see the path to be followed to obtain a component of this differential d_U. Let $D_{p,q}$ be a starting point. First you go up from $D_{p,q}$ following $g_{p,q}$ arriving at $C_{p,q}$. Then follow for example the ϕ_2 path of the above figure going from $C_{p,q}$ to $C_{p-2,q+2}$. Then again an arrow $d'_{p-2,q+2} : C_{p-2,q+2} \to C_{p-3,q+2}$. And finally get back in U_* by $f_{p-3,q+2} : C_{p-3,q+2} \to D_{p-3,q+2}$. This composition $f_{p-3,q+2}d'_{p-2,q+2}\phi_2 g_{p,q}$ is the arrow $d^3_{p,q} : D_{p,q} \to D_{p-3,q+2}$ of the multicomplex

$(D_{p,q}, d^r_{p,q})$.

$$\begin{tikzcd}
C_{p-3,q+2} \arrow[dd, "f"'] & C_{p-2,q+2} \arrow[l, "d'=\widehat{\delta}"'] & & \\
& C_{p-2,q+1} \arrow[u, "h"] & C_{p-1,q+1} \arrow[l, "d'=\widehat{\delta}"'] & \\
D_{p-3,q+2} & & C_{p-1,q} \arrow[u, "h"] & C_{p,q} \arrow[l, "d'=\widehat{\delta}"'] \arrow[uull, bend right, "\phi_2"'] \\
& & & D_{p,q} \arrow[u, "g"'] \arrow[ulll, "d^3_{p,q}"]
\end{tikzcd}$$

Do the same for every shift r and you obtain the multicomplex $(U_*, d^r_{p,q})$. The components of the final reduction $T_* \Rrightarrow U_*$ are quite analogous: every component starting from a C_{pq} or a $D_{p,q}$ and going to another one is made of a snake path plus a few simple components added at the departure and/or the arrival. ■

In what context this theorem can be used? We will see several different contexts where this theorem is a key tool. The simplest one is the following. If ever the $D_{p,q}$ are *effective*, the homology groups of U_* are elementarily computable. We will meet many cases where the "main" bicomplex is only locally effective, so that the homology groups of the totalization in general are not reachable. But frequently we can obtain for example the *effective homology* of every column. Our theorem will then give us an equivalence between the initial totalisation and another one coming from a multicomplex where the components are on the contrary *effective*. Then it will be possible to compute the homology groups.

Another classical use of the bicomplex spectral sequence theorem concerns the case where *every* column and row is "almost" exact. We will do the same in our constructive framework, obtaining a *constructive* result. Without any spectral sequence.

Theorem 76 — *Let $(C_{p,q}, d'_{p,q}, d''_{p,q})$ be a first quadrant bicomplex satisfying the following properties.*

- *Every (p)-column is exact except at $(p, 0)$ producing a homology group $H''_{p,0}$.*
- *Every (q)-row is exact except at $(0, q)$ producing a homology group $H'_{0,q}$.*
- *We assume* reductions *are available:*

$$(C_{p,*}, d'') \Rrightarrow H''_{p,0}; \qquad (C_{*,q}, d') \Rrightarrow H'_{0,q}.$$

Then an equivalence can be constructed*:*

$$(H''_{p,0}, d')_p \Lleftarrow\!\!\Rrightarrow (H'_{0,q}, d'')_q.$$

The statement of the theorem needs a few explanations. In the column direction for example, every colum is a chain-complex and requiring its exactness makes sense. The vertical exactness is required in any position (p, q) with $q > 0$. In position $(p, 0)$, the arrow $d''_{p,1} : C_{p,1} \to C_{p,0}$ is not necessarily surjective, which defines the homology group $H''_{p,0} = C_{p,0}/d''_{p,1}(C_{p,1})$. Now the horizontal arrow $d'_{p,0}$ induces a map $d'_{p,0} : H''_{p,0} \to H''_{p-1,0}$ and this produces a chain-complex $(H''_{p,0}, d')_p$. The same for the rows. The classical result obtained in this case is that the homology groups of both complexes $(H''_{p,0}, d')_p$ and $(H'_{0,q}, d'')_q$ are isomorphic. Here, using the reductions of the statement, we *construct* an equivalence between these complexes, which of course implies the isomorphism between homology groups.

PROOF. Let T_* be the totalisation of our bicomplex. Applying the Bicomplex Reduction Theorem produces a reduction: $T_* \Rrightarrow (H''_{p,0}, d')_p$. Doing the same with the rows finally gives:

$$(H''_{p,0}, d')_p \Lleftarrow T_* \Rrightarrow (H'_{0,q}, d'')_q$$

■

It is the first example where a natural *equivalence* is obtained, instead of a *reduction*. This is frequent.

We are finally ready to prove the *constructive* coherence of the Torsion groups.

Theorem 77 — *Let $\mathfrak{R} = \mathfrak{k}[x_1, \ldots, x_m]$ be the localized polynomial ring and M and N two $\mathfrak{R}$-modules. Let* $\mathrm{Rsl}(M)$ *and* $\mathrm{Rsl}(N))$ *be some* effective *free resolutions. An explicit equivalence can be installed between* $\mathrm{Rsl}_*(M) \otimes_{\mathfrak{R}} N$ *and* $M \otimes_{\mathfrak{R}} \mathrm{Rsl}_*(N)$.

PROOF. Reductions $\mathrm{Rsl}(M) \Rrightarrow M_*$ and $\mathrm{Rsl}(N) \Rrightarrow N_*$ are available. We consider the bicomplex $\mathrm{Rsl}(M) \otimes_{\mathfrak{R}} \mathrm{Rsl}(N)$. The Koszul convention implies the totalization actually is a chain-complex. If we examine the (p)-column, the left factor $\mathrm{Rsl}_p(M)$ of the tensor product $C_{p,q} = \mathrm{Rsl}_p(M) \otimes_{\mathfrak{R}} \mathrm{Rsl}_q(N)$ is independent of q. The $\mathfrak{R}$-module $\mathrm{Rsl}_p(M)$ is free of rank r_p so that the tensor product $\mathrm{Rsl}_p(M) \otimes_{\mathfrak{R}} \mathrm{Rsl}_*(N)$ is nothing but the direct sum of r_p copies of $\mathrm{Rsl}_p(N)$. In particular the reduction $\mathrm{Rsl}_*(N) \Rrightarrow N$ becomes a reduction $\mathrm{Rsl}_p(M) \otimes \mathrm{Rsl}_*(N) \Rrightarrow \mathrm{Rsl}_p(M) \otimes N$: the homology of every (p)-column is concentrated at $(p, 0)$. We are exactly in the situation of the previous theorem, obtaining an explicit equivalence:

$$\mathrm{Rsl}_*(M) \otimes_{\mathfrak{R}} N \Lleftarrow T_* \Rrightarrow M \otimes_{\mathfrak{R}} \mathrm{Rsl}_*(N).$$

■

Note the equivalence depends for example on the chosen isomorphisms $\mathrm{Rsl}_p(M) \cong \mathfrak{R}^{r_p}$. The homotopy operators of $\mathrm{Rsl}_*(N)$ in general *are not* $\mathfrak{R}$-morphisms and the expressions of the induced homotopy operator over $\mathrm{Rsl}_p(M) \otimes_{\mathfrak{R}} \mathrm{Rsl}_*(N)$ can so be modified: the splitting into r_p components *is not* intrinsic.

6 Effective homology of Koszul complexes.

6.1 Presentation.

The Koszul complexes play an important role when studying the *formal integrability problem* of PDE systems. The data in this case is a number of (independent) variables m, a ground field $\mathfrak{k} = \mathbb{R}$ or $\mathbb{C}$, the ring $\mathfrak{R} = \mathfrak{k}[x_1, \ldots, x_m]_0$ and an $\mathfrak{R}$-module of finite type M coming from the PDE system. The nature of the PDE system then strongly depends on the torsion groups $\mathrm{Tor}_*(M, \mathfrak{k})$ [28].

We explain in this Section how constructive homological algebra gives completely new methods to study this problem. Usually the torsion groups are computed as follows. First construct a finite free $\mathfrak{R}$-resolution of M, it is the classical Hilbert's syzygy problem. Efficient theoretical and concrete methods are available, but they are rather technical; the Groebner basis techniques are necessary. Then the tensor product $\mathrm{Rsl}(M) \otimes_{\mathfrak{R}} \mathfrak{k}$ is a finite chain-complex of finite dimensional $\mathfrak{k}$-vector spaces, the homology groups of which can be elementarily computed.

But it happens the theoretical result at the origin of this computation comes from the symmetric definition $\mathrm{Tor}_*(M, \mathfrak{k}) = H_*(M \otimes_{\mathfrak{R}} \mathrm{Ksz}(\mathfrak{k})) = H_*(\mathrm{Ksz}(M))$. If these torsion groups are sufficiently null, then the module M is *involutive*, which expresses that "good" coordinate systems can be used to study the algebraic nature of M. As usual, the homological condition allows the user to claim there *exists* good coordinate systems. Making *constructive* such a statement is a natural goal; if such constructive results are obtained, we can reasonably hope to be able to *concretely* use the nice results of [28].

To conveniently explain how our constructive methods can be used, we choose a framework a little simpler; the translation in the general framework is very easy. This framework is also chosen to allow us to give simple machine demonstrations with the current available Kenzo programs.

UOStated 78 — *In this section, the ground field $\mathfrak{k}$ is an arbitrary commutative field; in the Kenzo demonstrations, $\mathfrak{k} = \mathbb{Q}$. The ring $\mathfrak{R}$ is as before $\mathfrak{k}[x_1, \ldots, x_m]$. Instead of an $\mathfrak{R}$-module M, we consider an ideal $I = <g_1, \ldots, g_n> \subset \mathfrak{R}$ and the corresponding module $M = \mathfrak{R}/I$. We intend to construct a version* with effective homology *of Ksz(M) = Ksz$(\mathfrak{R}/I)$.*

The Groebner methods will play also an essential role, but with a completely different organization, significantly simpler and more conceptual from a theoretical point of view, at least when the general style of *constructive* homological algebra is understood.

6.2 Constructive homological algebra and short exact sequences of chain-complexes.

Theorem 22 explains how a short exact sequence of chain-complexes produces a long exact sequence of the corresponding homology groups. This exact sequence is implicitly assumed solving computational problems when you know the homology groups of two chain-complexes and you want to obtain the homology groups of the third one. Section 2.6.1 was devoted to elementary positive examples, but we saw later, Section 3.3.2, that in general exact sequences lead to *extension problems* which can be really difficult.

This section will replace the long exact sequence of a short exact sequence of chain-complexes by simple *constructive* results, which systematically avoid this difficulty. We will see why constructive homological algebra is also in particular a *general solving method* for extension problems.

We work in this subsection in a *quite general framework*. The ground ring $\mathfrak{R}$ is an arbitrary unitary commutative ring, and in fact its multiplicative structure is never used, it could be simply an Abelian group. No ground field is concerned, except if $\mathfrak{R}$ is itself a field...

We begin with an easy extension of the Cone Reduction Theorem 62.

Theorem 79 (Cone Equivalence Theorem) — *Let $\phi : C_{*,EH} \leftarrow C'_{*,EH}$ be a chain-complex morphism between two chain-complexes* with effective homology*. Then a general algorithm computes a version* with effective homology $\mathrm{Cone}(\phi)_{EH}$ *of the cone.*

$$\begin{tikzcd}
& \widehat{C}_* \arrow[loop left, "\ell h"] \arrow[loop right, "rh"] \arrow[dl, shift right, "\ell f"'] \arrow[dr, shift left, two heads, "rf"] & & & \widehat{C}'_* \arrow[lll, bend left, "\widehat{\phi}"'] \arrow[loop left, "\ell h'"] \arrow[loop right, "rh'"] \arrow[dl, shift right, "\ell f'"'] \arrow[dr, shift left, two heads, "rf'"] & \\
C_* \arrow[ur, shift right, "\ell g"'] & & EC_* \arrow[ul, shift left, "rg"] & C'_* \arrow[lll, bend left, "\phi"] \arrow[ur, shift right, "\ell g'"'] & & EC'_* \arrow[ul, shift left, "rg'"] \arrow[llll, bend left, "E\phi"]
\end{tikzcd}$$

Proof. We start with two equivalences $C_* \lll \widehat{C}_* \ggg EC_*$ and $C'_* \lll \widehat{C}'_* \ggg EC'_*$ and the morphism $\phi : C_* \leftarrow C'_*$. In the figure above, ℓ = left and r = right, this for each given equivalence. The morphism ϕ naturally induces "parallel" morphisms $\widehat{\phi} := (\ell g)\,\phi\,(\ell f') : \widehat{C}_* \leftarrow \widehat{C}'_*$ and then $E\phi := (rf)\,(\ell g)\,\phi\,(\ell f')\,(rg') : EC_* \leftarrow EC'_*$.

As usual we can consider ϕ is a perturbation of the differential of $\mathrm{Cone}(C_* \overset{0}{\leftarrow} C'_*)$. Using the Easy Perturbation lemma 4.8.2 produces a reduction $\mathrm{Cone}(\phi) \lll \mathrm{Cone}(\widehat{\phi})$ where in fact the morphism $\widehat{\phi}$ is *produced* by the lemma. Applying in the same way the Basic Perturbation lemma between $\mathrm{Cone}(\widehat{C}_* \overset{0}{\leftarrow} \widehat{C}'_*)$ and $\mathrm{Cone}(\widehat{C}_* \overset{\widehat{\phi}}{\leftarrow} \widehat{C}'_*)$ produces in turn a new reduction $\mathrm{Cone}(\widehat{\phi}) \ggg \mathrm{Cone}(E\phi)$ where again, the morphism $E\phi$ is in fact *produced* by the BPL. Combining these reductions gives the looked-for equivalence:

$$\mathrm{Cone}(C_* \overset{\phi}{\leftarrow} C'_*) \lll \mathrm{Cone}(\widehat{C}_* \overset{\widehat{\phi}}{\leftarrow} \widehat{C}'_*) \ggg \mathrm{Cone}(EC_* \overset{E\phi}{\leftarrow} EC'_*)$$

∎

You see how our perturbation lemmas are used. Some process is applied to the left hand term of an equivalence, here the cone construction. This process induces something analogous over the central chain-complexe of the given equivalence, thanks to the easy perturbation lemma. The left hand reduction is not here modified, this reduction is only used to *copy* the perturbation into the central chain complex. Then the actual Basic Perturbation lemma is applied to take account of the perturbation in the central chain complex to replace the right hand reduction by a new appropriate reduction; in general the differential of the right hand chain complex is modified.

Definition 80 — An *effective short exact sequence* of chain-complexes is a diagram:

$$0 \xleftarrow{0} A_* \underset{j}{\overset{\sigma}{\rightleftarrows}} B_* \underset{i}{\overset{\rho}{\rightleftarrows}} C_* \longleftarrow 0$$

where i and j are *chain-complex* morphisms, ρ (retraction) and σ (section) are *graded module* morphisms satisfying:

- $\rho i = \mathrm{id}_{C_*}$;
- $i\rho + \sigma j = \mathrm{id}_{B_*}$;
- $j\sigma = \mathrm{id}_{A_*}$.

It is an exact sequence in both directions, but to the left it is an exact sequence of *chain-complexes*, the exact sequence we are mainly interested in, and to the right it is only an exact sequence of *graded modules*, no compatibility in general with the differentials. The components ρ and σ are nothing but a homotopy operator describing a *reduction* to 0 of our "total" chain-complex: you can think of this exact sequence as a bicomplex with only three columns non-null. As usual for the homotopy operators, weak properties are only required, and here for example it is not required ρ and σ are compatible with the differentials. Otherwise the chain-complex B_*, *differential included*, would be the direct sum of A_* and C_*, a trivial situation without any interest. The exactness expresses i is injective, j is surjective, and ρ and σ define a sum decomposition $B_* = \operatorname{im} i \oplus \ker \rho = \operatorname{im} \sigma \oplus \ker j$, but this decomposition *is not* in general a subcomplex decomposition, making the hoped-for results non-trivial.

Theorem 81 (SES Theorems) — *Let*

$$0 \xleftarrow{0} A_* \underset{j}{\overset{\sigma}{\rightleftarrows}} B_* \underset{i}{\overset{\rho}{\rightleftarrows}} C_* \longleftarrow 0$$

be an effective *short exact sequence of chain-complexes. Then three general algorithms are available:*

$$\begin{aligned}
\mathrm{SES}_1 &: (B_{*,EH}, C_{*,EH}) \mapsto A_{*,EH}\\
\mathrm{SES}_2 &: (A_{*,EH}, C_{*,EH}) \mapsto B_{*,EH}\\
\mathrm{SES}_3 &: (A_{*,EH}, B_{*,EH}) \mapsto C_{*,EH}
\end{aligned}$$

producing a version with effective homology *of one chain-complex when versions* with effective homology *of both others are given.*

SES = Short Exact Sequence. Observe the process is perfectly *stable*: the *type* of the result is exactly the same as for the given objects. The obtained object can then be used later in another exact or spectral sequence, and so on.

PROOF.

Let us begin with the SES_1 case.

Lemma 82 — *The effective exact sequence produces a reduction: Cone*$(i) \Rrightarrow A_*$.

PROOF. It is again a simple application of BPL. We mentioned, when describing the notion of *effective* short exact sequence, that ρ and σ are "weak" morphisms. This negative property is nomore an obstacle if we cancel the differentials of our three chain-complexes. Let us call A_*^0, B_*^0, C_*^0 these chain-complexes with null differentials. It is easy to obtain the looked-for reduction in this simple case. It is:

$$\rho^0 = (f^0, g^0, h^0) : \mathrm{Cone}(i : B_*^0 \leftarrow C_*^0) \Rrightarrow A_*^0$$

The morphism $f^0 : \mathrm{Cone}(i) \to A_*^0$ is the projection defined by $j : A_*^0 \leftarrow B_*^0$, null on the C_*^0 component of the cone. The morphism g^0 is defined by the section σ with values in the B_*^0 component of the cone. Finally the homotopy operator h^0 is the retraction $\rho : B_*^0 \to C_*^0$ inside the cone. The reduction properties are direct consequence of the relations satisfied by i, j, ρ and σ. Note the *components* of our new cone have null differentials, but the cone itself has the component i non null except if $C_* = 0$.

Now we reinstall the right differentials *over the cone.* Two components for the perturbation $\widehat{\delta}$, a differential in general non trivial over B_* and another one over C_*. Combined with the initial homotopy operator of our reduction, we see $(h^0\widehat{\delta})^2$ is null. The nilpotency condition is satisfied.

Using Shih's formula for the new reduction, we obtain the reduction:

$$\rho = (f, g, h) : \mathrm{Cone}(i : B_* \leftarrow C_*) \Rrightarrow A_*$$

with $f = f^0 = j$ and $h = h^0 = \rho$ not modified, but with $g = \sigma - \rho d_{B_*}\sigma$. Furthermore, the new differential to install on the small chain-complex is by chance the initial differential d_{A_*} of A_*. ∎

PROOF OF THEOREM CONTINUED. Consider the sequence:

$$A_* \Lleftarrow \mathrm{Cone}(i) \Lleftarrow \mathrm{Cone}(\widehat{i}) \Rrightarrow \mathrm{Cone}(Ei).$$

The central and the right hand reductions are produced by the Cone Equivalence Theorem, using the available equivalences describing the chain-complexes B_* and C_* as chain-complexes with effective homology. The left hand reduction is produced by the lemma just proved. Composing the central and the left hand reductions gives another reduction and an equivalence between A_* and $\mathrm{Cone}(Ei)$ is obtained, describing also A_* as a chain-complex with effective homology.

The case SES_3 is symmetric and left to the reader.

Let us finally consider the case SES_2, different.

Lemma 83 — *The* effective *short exact sequence generates a connection* chain-complex *morphism* $\chi : A_* \to C_*^{[1]}$.

The "exponent " [1] explains the *suspension functor* is applied to the chain-complex C_*: the degree of an element is increased by 1 and the differential is replaced by the opposite.

PROOF. The connection morphism is defined as the composition $\chi = \rho d\sigma$ where the differential cannot be something else than $d = d_B$; this differential has degree -1 and is the cause of the suspension. We must verify the compatibility of this claimed *chain-complex* morphism with the differentials of A_* and $C_*^{[1]}$.

Let us consider an element $a \in A_n$, then its lifting σa in B_*, and let us try to use $d_B d_B = 0$ and also $\sigma j + i\rho = \mathrm{id}$. First:

$$\begin{array}{rclr} d\sigma a & = & \sigma j d\sigma a + i\rho d\sigma a & (\sigma j + i\rho = \mathrm{id}) \\ & = & \sigma da + i\rho d\sigma a & (jd = dj \text{ and} j\sigma = \mathrm{id}) \end{array}$$

Let us apply again d_B:

$$\begin{array}{rclr} 0 & = & d\sigma da + di\rho d\sigma a & \\ & = & \sigma j d\sigma da + i\rho d\sigma da + \sigma j di\rho d\sigma a + i\rho di\rho d\sigma a & (\sigma j + i\rho = \mathrm{id}) \\ & = & 0 + i\rho d\sigma da + 0 + id\rho d\sigma a & \end{array}$$

for $jd = dj$, $j\sigma = \mathrm{id}$, $dd = 0$, $jd = dj$ again and $ji = 0$. The morphism i is injective, which implies:

$$d(\rho d\sigma)a = -(\rho d\sigma)(da).$$

∎

This looks a little magic, but in fact, as in ordinary magic, there is an explanation. The central B_* is, *as graded module*, the direct sum of A_* and C_*. If you think of an element of B_* as having two components, one in A_* and the other one in C_*, then you obtain an expression of the differential of d_{B_*} as working in $A_* \oplus C_*$; the differential is a 2×2 matrix of maps, the component $C_* \to A_{*-1}$ being null because i and j are compatible with the differentials and $ji = 0$; the component $A_* \to C_*$ is our connection map. We so obtain a cone diagram:

$$\begin{tikzcd} \cdots & A_{n-2} \arrow[l, "d"'] \arrow[dl, "\chi"'] & A_{n-1} \arrow[l, "d"'] \arrow[dl, "\chi"'] & A_n \arrow[l, "d"'] \arrow[dl, "\chi"'] & \cdots \arrow[l, "d"'] \arrow[dl, "\chi"'] \\ \cdots & C_{n-2} \arrow[l, "d"'] & C_{n-1} \arrow[l, "d"'] & C_n \arrow[l, "d"'] & \cdots \arrow[l, "d"'] \end{tikzcd}$$

The total differential of this diagram is null if and only if every parallelogram is anticommutative.

Proof of Theorem continued. The previous diagram also explains in fact B_* is canonically isomorphic to Cone(χ). Using the Cone Equivalence Theorem and the equivalences describing the effective homology of A_* and B_*, we obtain the looked-for equivalence between B_* and an effective chain-complex. ∎

6.3 Solution for monomial ideals.

We come back to our goal in commutative algebra: computing the *effective* homology of $\mathrm{Ksz}(\mathfrak{R}/I)$ for I an ideal of $\mathfrak{R} = \mathfrak{k}[x_1, \ldots, x_m]_0$. Our ring is Noetherian and I is described by a finite set of generators $I = {<}g_1, \ldots, g_n{>}$. Our work is decomposed in three steps:

1. Using a Groebner basis, we replace I by I' a *monomial* ideal to be considered as a *good simple* approximation of I;
2. A recursive process over I' using a number of times the BPL gives a simple solution for I';
3. Applying again the BPL between I and I' will give the solution for the ideal I.

Step 1 is standard. You choose a coherent monomial order over $\mathfrak{R}$, then a reduced Groebner basis is canonically defined for our ideal I. We assume our expression $I = {<}g_1, \ldots, g_n{>}$ just uses this Groebner basis.

The ideal I' is obtained by replacing every generator g_i by its *leading term* g'_i: $I' := {<}g'_1, \ldots, g'_n{>}$. This process is interesting for two reasons:

- The monomial ideal I', because it is monomial, is more comfortable.
- Both ideals I and I' are "close" to each other: the *graded modules* $\mathfrak{R}/I$ and $\mathfrak{R}/I'$ are *canonically isomorphic.*

Of course the multiplicative structures of $\mathfrak{R}/I$ and $\mathfrak{R}/I'$ are different, but the isomorphism between the underlying *graded modules* will be enough when applying the BPL to process this difference.

In the rest of this section, we assume our ideal I is monomial: every generator g_i of $I = {<}g_1, \ldots, g_n{>}$ is a monomial of $\mathfrak{R} = \mathfrak{k}[x_1, \ldots, x_m]_0$.

The recursive process then consists in obtaining the result for the simpler ideal $J = {<}g_2, \ldots, g_n{>}$, the generator g_1 being removed. What about the exact nature of the relation between I and J? We must use the notion of *quotient* of two ideals; the quotient $I_1 : I_2$ of two ideals I_1 and I_2 is $(I_1 : I_2) := \{a \in \mathfrak{R} \ \textbf{st} \ aI_2 \subset I_1\}$.

Proposition 84 — *An ideal $I = {<}g_1, \ldots, g_n{>} \subset \mathfrak{R}$ produces an* effective *short exact sequence of $\mathfrak{R}$-modules:*

$$0 \leftarrow \frac{\mathfrak{R}}{{<}g_1, \ldots, g_n{>}} \leftarrow \frac{\mathfrak{R}}{{<}g_2, \ldots, g_n{>}} \leftarrow \frac{\mathfrak{R}}{{<}g_2, \ldots, g_n{>}:{<}g_1{>}} \leftarrow 0.$$

PROOF. Exercise. ∎

In this exercise, please observe the initial monomorphism $\mathfrak{R}/{<}g_2,\ldots,g_n{>} \leftarrow \mathfrak{R}/({<}g_2,\ldots,g_n{>}:{<}g_1{>})$ is defined by the *multiplication by* g_1, while the terminal epimorphism $\mathfrak{R}/{<}g_1,\ldots,g_n{>} \leftarrow \mathfrak{R}/{<}g_2,\ldots,g_n{>}$ is the canonical projection; this remark will be important later. These monomorphism and epimorphism are $\mathfrak{R}$-module morphisms. To make effective the exact sequence, a section σ and a retraction ρ are needed, see Definition 80. The section is the "brute" lifting which sends a non-null monomial – in fact its class modulo the ideal – to the same. The retraction examines whether a monomial is divisible by g_1; if yes the retraction gives the quotient by g_1, otherwise the result is null. These section and retraction are $\mathfrak{k}$-linear but not at all $\mathfrak{R}$-morphisms.

Proposition 85 — *If the given generators of* ${<}g_1,\ldots,g_n{>}$ *are* monomials, *then* ${<}g_2,\ldots,g_n{>}:{<}g_1{>} = {<}g'_2,\ldots,g'_n{>}$ *with* $g'_i = \operatorname{lcm}(g_1,g_i)/g_1$ *for* $i \geq 2$.

PROOF. Exercise. ∎

Which implies if, thanks to the short exact sequence, a recursive process reduces some work for $\mathfrak{R}/{<}g_1,\ldots,g_n{>}$ to the analogous work for $\mathfrak{R}/{<}g_2,\ldots,g_n{>}$ and $\mathfrak{R}/{<}g'_2,\ldots,g'_n{>}$, there remains to *start* the recursive process.

Corollary 86 — *A general algorithm computes:*

$$\left[\operatorname{Ksz}\left(\frac{\mathfrak{R}}{{<}g_2,\ldots,g_n{>}}\right)_{EH}, \operatorname{Ksz}\left(\frac{\mathfrak{R}}{{<}g'_2,\ldots,g'_n{>}}\right)_{EH}\right] \mapsto \operatorname{Ksz}\left(\frac{\mathfrak{R}}{{<}g_1,g_2,\ldots,g_n{>}}\right)_{EH}$$

when the generators $g_1,g_2,\ldots,g_n$ *are monomials, when* $g'_i = \operatorname{lcm}(g_1,g_i)/g_1$*, where* $\operatorname{Ksz}(\cdots)_{EH}$ *is a version* with effective homology *of the Koszul complex* $\operatorname{Ksz}(\cdots)$.

PROOF. The constructor $M \mapsto \operatorname{Ksz}(M)$ is a functor from $\mathfrak{R}$-modules to chain-complexes. An $\mathfrak{R}$-module morphism $f : M \to N$ generates a chain-complex morphism $f := \operatorname{Ksz}(f) : \operatorname{Ksz}(M) \to \operatorname{Ksz}(N)$. Applying this functor, the *effective* short exact sequence of $\mathfrak{R}$-*modules* of Proposition 84 becomes an *effective* short exact sequence of *chain-complexes*:

$$0 \leftarrow \operatorname{Ksz}\left(\frac{\mathfrak{R}}{{<}g_1,\ldots{>}}\right)_{EH} \leftarrow \operatorname{Ksz}\left(\frac{\mathfrak{R}}{{<}g_2,\ldots{>}}\right)_{EH} \leftarrow \operatorname{Ksz}\left(\frac{\mathfrak{R}}{{<}g'_2,\ldots{>}}\right)_{EH} \leftarrow 0$$

Applying the SES_1 case of Theorem 81 gives the result. ∎

We noted in Proposition 84 the section σ for example is only k-linear; then $\operatorname{Ksz}(\sigma)$ is defined but is not compatible with differentials; it is only a graded-module morphism.

The recursive process is now installed: computing the effective homology of a Koszul complex $\operatorname{Ksz}(\mathfrak{R}/{<}g_1,\ldots,g_n{>})$ is reduced to two analogous problems with one generator less. What about the starting point of this induction? The minimal case is 0 generator; we must determine $\operatorname{Ksz}(\mathfrak{R}/{<}{>})_{EH} = \operatorname{Ksz}(\mathfrak{R})_{EH}$. This was done at Theorem 70, which theorem was a translation of Theorem 67. Combining this remark with the above corollary gives the main result of this section.

Theorem 87 — *A general algorithm computes:*

$$<g_1, \ldots, g_n> \mapsto \mathrm{Ksz}\left(\frac{\mathfrak{R}}{<g_1, \ldots, g_n>}\right)_{EH}$$

where $g_1, \ldots, g_n$ are monomial *generators in our localized polynomial ring $\mathfrak{R}$.* ■

The *homological problem* for the chain-complex $\mathrm{Ksz}(\mathfrak{R}/<g_1, \ldots, g_n>)$ is solved in the *monomial case*. How to obtain the same result in the general case?

6.4 Installing a general multigrading.

It was explained at the beginning of the previous section we intend to apply again the BPL to process the difference between an arbitrary ideal I and its monomial approximation I'. The required nilpotency hypothesis needs a careful use of monomial orders. Two ingredients are necessary.

On one hand we must *delocalize* the problem, replacing the localized ring $\mathfrak{R} = \mathfrak{k}[x_1, \ldots, x_m]_0$ by the ordinary polynomial ring $\overline{\mathfrak{R}} = \mathfrak{k}[x_1, \ldots, x_m]$. We will prove later that if I is an ideal of $\mathfrak{R}$, then $H_*(\mathrm{Ksz}_{\mathfrak{R}}(\mathfrak{R}/I)) \cong H_*(\mathrm{Ksz}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}}/\overline{I}))$ if $\overline{I} = I \cap \overline{\mathfrak{R}}$, so that instead of studying the problem of I inside $\mathfrak{R}$, we can study the case of $\overline{I}$ and $\overline{\mathfrak{R}}$; furthermore this isomorphism between different homology groups will be *constructive*, and a solution for the homological problem of $\mathrm{Ksz}_{\mathfrak{R}}(\mathfrak{R}/I)$ is equivalent to a solution for $\mathrm{Ksz}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}}/\overline{I})$: we can get rid of the denominators.

Definition 88 — An $\overline{\mathfrak{R}}$-ideal $\overline{I} \subset \overline{\mathfrak{R}}$ is *localized at* $0 \in \mathfrak{k}^m$ if $(\overline{I}\mathfrak{R}) \cap \overline{\mathfrak{R}} = \overline{I}$.

In this definition $\overline{I}\mathfrak{R}$ is the $\mathfrak{R}$-ideal generated by $\overline{I} \subset \overline{\mathfrak{R}} \subset \mathfrak{R}$. If I is an ideal of the local ring $\mathfrak{R}$, then $\overline{I} = I \cap \overline{\mathfrak{R}}$ is an $\overline{\mathfrak{R}}$-ideal localized at 0, and all those ideals are obtained in this way. The inclusion $\overline{I} \subset (\overline{I}\mathfrak{R}) \cap \overline{\mathfrak{R}}$ is always satisfied, but the ideal $\overline{I} = <1 - x> \subset \mathfrak{k}[x]$ is not localized at 0, for $(\overline{I}\mathfrak{R}) \cap \overline{\mathfrak{R}} = \overline{\mathfrak{R}} \neq \overline{I}$.

On the other hand, in order to be able to use the Groebner techniques in our context, we must define and handle carefully multigradings and monomial orders. Once for all, we choose a *Groebner monomial order*. If $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ is a monomial, its *multigrading* is an m-tuple $\mu(x_1^{\alpha_1} \cdots x_m^{\alpha_m}) := [\alpha_1, \ldots, \alpha_m]$. We consider an ideal $\overline{I} \subset \overline{\mathfrak{R}}$ defined by a reduced Groebner basis $<g_1, \ldots, g_n>$ for the chosen monomial order. The leading term of g_i is g_i', a monomial, and a canonical $\mathfrak{k}$-vector space isomorphism is defined between $\overline{\mathfrak{R}}/\overline{I}$ and $\overline{\mathfrak{R}}/\overline{I}'$ if $\overline{I}' = <g_1', \ldots, g_n'>$.

The ideal $\overline{I}'$ is monomial and $\overline{\mathfrak{R}}/\overline{I}'$ is multigraded. The Koszul complex $\mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I}')$ is also multigraded if we decide:

$$\mu(x_1^{\alpha_1} \cdots x_m^{\alpha_m}\, dx_1^{\beta_1} \cdots dx_m^{\beta_m}) := [\alpha_1 + \beta_1, \cdots, \alpha_m + \beta_m].$$

where $\alpha_i \in \mathbb{N}$ and $\beta_i \in \{0, 1\}$. In particular the differential of the Koszul complex is multigraded: a differential is made of terms where a dx_i is replaced by a x_i, which does not change the multigrading.

The work of the previous section for Koszul complexes of monomial ideals can be repeated without any change for the case of $\overline{\mathfrak{R}}$ and $\overline{I}'$ in the present section instead of $\mathfrak{R}$ and I in the previous section. In particular we must use the initial reduction $(f, g, h) : \mathrm{Ksz}(\overline{\mathfrak{R}}) \Rrightarrow \mathfrak{k}_*$ defined exactly in the same way. Note the three components of the reduction are also multigraded: the components f and g are trivial except for elements of null multigradation; and the homotopy operator h, see its detailed construction at page 62, does the contrary of the differential: every term is obtained by replacing some x_i by the corresponding dx_i. Using an obvious terminology, we can state:

Proposition 89 — *The reduction* $\mathrm{Ksz}(\overline{\mathfrak{R}}) \Rrightarrow \mathfrak{k}_*$ *constructed as in Section 5.6 is multigraded.*

Note in particular taking or not the denominators *does not* "significantly" change the effective homology of the Koszul complex of the ground ring.

For a monomial ideal $\overline{I}'$, we must apply a few times the Cone Reduction Theorem 62 to compute the effective homology of the corresponding module, as explained in the previous section. We need multigraded versions of the Basic Perturbation lemma and its applications, in particular for the Cone Reduction Theorem.

Theorem 90 — *If the data* $\rho = (f, g, h) : (\widehat{C}_*, \widehat{d}) \Rrightarrow (C_*, d)$ *and* $\widehat{\delta} : \widehat{C}_* \to \widehat{C}_{*-1}$ *of the Basic Perturbation lemma 50 are multigraded, the resulting reduction* $\rho' = (f', g', h') : (\widehat{C}_*, \widehat{d} + \widehat{\delta}) \Rrightarrow (C_*, d + \delta)$ *is also multigraded.*

Proof. In this statement, the underlying chain-complexes are multigraded, and the various *given* operators respect the multigrading. The theorem asserts the same for the new reduction. Given the explicit formulas for the components of the new reduction, the proof is obvious. ∎

In the same way, if a morphism $\phi : C_* \leftarrow C'_*$ is a multigraded morphism between multigraded chain-complexes, and if the effective homology of both complexes is given and is multigraded too, then the effective homology of $\mathrm{Cone}(\phi)$ computed by Theorem 62 is also multigraded: the three components of both reductions describing the effective homology of the cone are multigraded, their source and target as well.

Remember the main step when computing the effective homology of $\mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I}')$ consists in using the *effective* short exact sequence of $\overline{\mathfrak{R}}$-modules:

$$0 \leftarrow \frac{\overline{\mathfrak{R}}}{<g'_1, \ldots, g'_n>} \xleftarrow{\mathrm{pr}} \frac{\overline{\mathfrak{R}}}{<g'_2, \ldots, g'_n>} \xleftarrow{\times g'_1} \frac{\overline{\mathfrak{R}}}{<g'_2, \ldots, g'_n>:<g'_1>} \leftarrow 0.$$

The generators are monomials, so that the epimorphism 'pr', the canonical projection, is multigraded. The monomorphism '$\times g'_1$' is the multiplication by a monomial, it is also multigraded if you *shift the multigrading* of the initial module $\overline{\mathfrak{R}}/(<g'_2, \ldots, g'_n>:<g'_1>)$ by $\mu(g'_1)$.

Starting from the *multigraded* effective homology $\mathrm{Ksz}(\overline{\mathfrak{R}}) \Rrightarrow \mathfrak{k}$, applying repetitively this process produces a version with *multigraded* effective homology of the Koszul complex of our monomial module:

$$\mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I}') \Lleftarrow \widehat{C}_* \Rrightarrow EC_*$$

where the three modules are multigraded, and the six reduction components as well. The following theorem is proved.

Theorem 91 — *An algorithm computes:*

$$\overline{I}' \mapsto [Ksz(\overline{\mathfrak{R}}/\overline{I}') \Lleftarrow \widehat{C}_* \Rrightarrow EC_*]$$

where $\overline{I}'$ is a monomial ideal of $\overline{\mathfrak{R}} = \mathfrak{k}[x_1, \ldots, x_m]$ and the result is a multigraded equivalence between the corresponding Koszul complex and an effective multigraded chain-complex of finite-dimensional $\mathfrak{k}$-vector spaces. ■

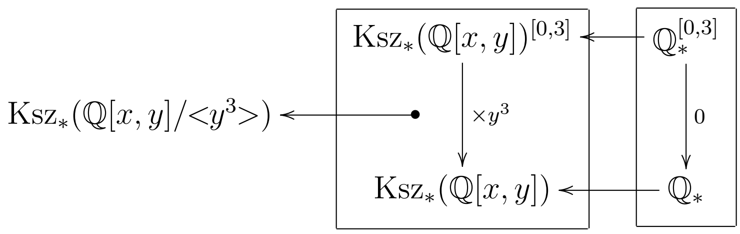
Example. We consider the toy example $\overline{I}' = <x^2, y^3> \subset \overline{\mathfrak{R}} = \mathbb{Q}[x, y]$; effective homology must in particular compute homology groups for the Koszul complex with representants for homology classes. The recursive process will lead to consider $\overline{I}'_0 = <>$, then $\overline{I}'_1 = <y^3>$ and finally $\overline{I}'$.

The result for $\overline{I}'_0$ was obtained at Theorem 70. The Koszul complex $\mathrm{Ksz}_*(\overline{\mathfrak{R}})$ is an $\overline{\mathfrak{R}}$-resolution of $\mathfrak{k} = \mathbb{Q}$ and its effective homology is a diagram:

$$\mathrm{Ksz}_*(\mathbb{Q}[x, y]) \overset{=}{\Lleftarrow} \mathrm{Ksz}_*(\mathbb{Q}[x, y]) \overset{\rho}{\Rrightarrow} \mathbb{Q}_*.$$

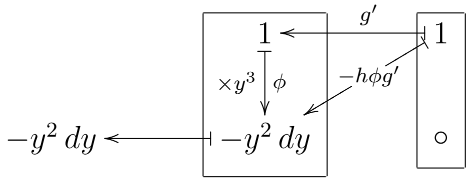
with $\mathbb{Q}_*$ the chain-complex with only $\mathbb{Q}$ in degree 0. Only one homology group, in degree 0, isomorphic to $\mathbb{Q}$; the representant of a generator is obtained by taking the image of the generator of $\mathbb{Q}_*$ in $\mathrm{Ksz}_*(\mathbb{Q}[x, y])$, it is the "base point" of this Koszul complex, namely $1 \in \mathrm{Ksz}_0(\mathbb{Q}[x, y])$.

The next figure extracts the important parts of the effective homology of $\mathrm{Ksz}_*(\mathbb{Q}[x, y]/<y^3>)$ when looking for a representant of the generator of the homology in degree 1.



The boxes are cone chain-complexes produced by Corollary 86. The null morphism between both copies of $\mathbb{Q}_*$ is the image of '$\times y^3$' between both copies of $\mathrm{Ksz}_*(\mathbb{Q}[x, y])$. The right hand box is the *effective* chain-complex describing the homology of $\mathrm{Ksz}_*(\mathbb{Q}[x, y]/<y^3>)$. The central box settles the necessary connection between the right hand box and $\mathrm{Ksz}_*(\mathbb{Q}[x, y]/<y^3>)$. The exponents $[0, 3]$ show the multigrading shift when necessary; in this way the '$\times y^3$' map is multigraded.

The next diagram displays at the right place elements of the nodes of the previous diagram.

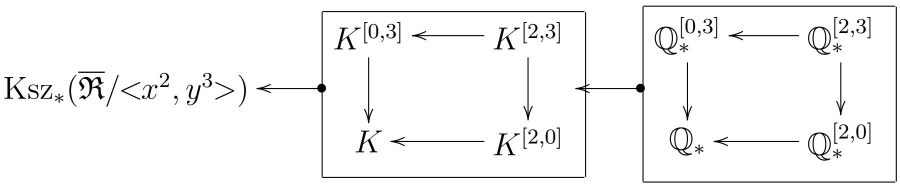


The right hand 1 is the generator of the “abstract” homology in degree 1. Its image in the intermediate box is obtained as explained in the cone reduction diagram of page 58: we have indicated in the present diagram the relevant arrows labelled ϕ, g' and $-h\phi g'$ of the generic cone diagram. Note in particular the role of the contraction h when obtaining the component $-y^2\,dy$. The conclusion is : a generator of the homology in degree 1 is the cycle $-y^2\,dy \in Z_1(\mathrm{Ksz}_*(\mathbb{Q}[x,y]/\langle y^3\rangle))$.

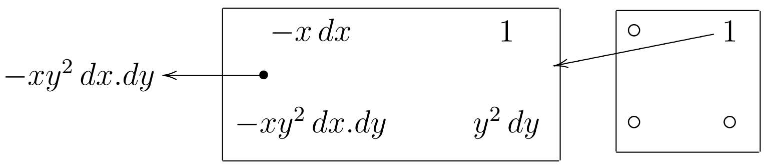
Now we must use the next short exact sequence to take care of the generator x^2 of the ideal $\overline{I}'$:

$$0 \leftarrow \frac{\overline{\mathfrak{R}}}{\langle x^2, y^3\rangle} \xleftarrow{\text{pr}} \frac{\overline{\mathfrak{R}}}{\langle y^3\rangle} \xleftarrow{\times x^2} \frac{\overline{\mathfrak{R}}}{\langle y^3\rangle} \leftarrow 0.$$

For in this case, $\langle y^3\rangle{:}\langle x^2\rangle = \langle y^3\rangle$. The available work above, combined with the Cone Reduction Theorem gives the effective homology of $\mathrm{Cone}(\times x^2)$ when applied to the corresponding Koszul complexes. Which cone can be reduced over $\mathrm{Ksz}_*(\overline{\mathfrak{R}}/\langle x^2, y^3\rangle)$. The main components of the result are in the diagram:



where K is a shorthand for $\mathrm{Ksz}_*(\overline{\mathfrak{R}})$. The cones we have to work with are now *cones of cones*, which explains why the boxes representing these cones are now square boxes; because of the recursive organization, the tower of cones can have in the general case an arbitrary number of floors. Playing here the same game as before for the homology generator in degree 2 leads to the diagram:



Please try to do it by hand; it is not so hard but in particular when you have to mix which has been done at the previous level for $\langle y^3\rangle$ with the new equivalence to be constructed, things become quickly relatively complex. And if you have an ideal with many generators, of course a machine program is necessary.

The Kenzo program can process these calculations. Let us make Kenzo construct the previous diagram. First we define the ideal. Every generator $x^\alpha y^\beta$ is coded as the integer list $(\alpha\ \beta)$ and the ideal as a list of generators.

```
> (setf I '((2 0) (0 3))) ✠
((2 0) (0 3))
```

Constructing the corresponding Koszul complex.

```
> (setf K (k-complex/i 2 I)) ✠
[K3 Chain-Complex]
```

Constructing the effective homology of the Koszul complex, assigned to the symbol EH.

```
> (setf EH (efhm K)) ✠
[K98 Equivalence K3 <= K71 => K74]
```

Kenzo automatically organizes the recursion process and returns an *equivalence* between the Koszul complex K3 $= \mathrm{Ksz}_*(\overline{\mathfrak{R}}/<x^2, y^3>)$ and the *effective* chain-complex K74 via another chain-complex K71, only *locally effective*, namely the left hand square of the above diagram.

The chain-complex K74 is effective and we can ask for the basis for example in degree 2.

```
> (basis (k 74) 2) ✠
(<Con1 <Con1 Z-GNRT>>)
```

The rank is 1 and the unique generator is in a cone of cones. Because this complex is effective, the homology groups are elementarily computed, for example in degree 2. The function homology-gen returns a list of generators for this homology.

```
> (homology-gen (k 74) 2)
(
----------------------------------------------------------------------{CMBN 2}
<1 * <Con1 <Con1 Z-GNRT>>>
------------------------------------------------------------------------------
)
```

Only one generator, we find again the same generator presented as a combination of degree 2 ({CMBN 2}) with one term, coefficient 1, and generator <Con1 <Con1 Z-GNRT>>.

Now, important intermediate step, we want to lift this generator of "abstract homology" into K71. We first extract this generator from the one element list, then apply the rg component (rg = right hand g) of our equivalence. The Lisp symbol '*' points to the last result returned.

```
> (first *) ✠
------------------------------------------------------------------------{CMBN 2}
<1 * <Con1 <Con1 Z-GNRT>>>
------------------------------------------------------------------------------
> (rg EH *) ✠
------------------------------------------------------------------------{CMBN 2}
<-1 * <Con0 <Con0 ((1 2) (1 1))>>>
<1 * <Con0 <Con1 ((0 2) (0 1))>>>
<-1 * <Con1 <Con0 ((1 0) (1 0))>>>
<1 * <Con1 <Con1 ((0 0) (0 0))>>>
------------------------------------------------------------------------------
```

You easily recognize the element which was displayed in the left hand box of the last diagram, on condition that you know how to translate for example `((1 2) (1 1))` $= xy^2\, dx.dy$. There remains to go to our Koszul complex, applying this time the `lf` component of the equivalence.

```
> (lf EH *) ✠
------------------------------------------------------------------------{CMBN 2}
<-1 * ((1 2) (1 1))>
------------------------------------------------------------------------------
```

The representant cyle is $-xy^2\, dx.dy$. Computing the *ordinary* homology of such a simple Koszul complex is elementary; computing the *effective* homology is already not so easy, think of the six morphisms defining the equivalence `K98` between our Koszul complex `K3` and the effective chain-complex `K74`; think also of the right differential to be installed in the cones of cones. For an ideal with more variables and more generators, this cannot reasonably be obtained without a machine. Let us for example consider the ideal $\overline{I}' = <v^3w^3x^3z^2,\ v^2w^3xyz^3, vw^3x^3yz^2, vw^3x^2y^2z^2, v^3w^3xy^2z, v^3w^3x^3y, w^2x^3y^3z^2, v^2w^3x^2y^3, v^2wxy^3z^3, v^2x^3y^2z^3, v^2w^2x^2y^2z^3, v^3w^2x^3yz^3>$ of $\mathbb{Q}[v,w,x,y,z]$. Working exactly as before, we use Kenzo to construct the ideal, the corresponding Koszul complex and its effective homology. The 3-homology has rank 9 and we extract the abstract generator number 7, for which a representant cycle is computed in the Koszul complex. You may observe the numerical notation of monomials by number lists is quickly more readable than the usual one.

```
> (setf I '((3 3 3 0 2) (2 3 1 1 3) (1 3 3 1 2) (1 3 2 2 2)
            (3 3 1 2 1) (3 3 3 1 0) (0 2 3 3 2) (2 3 2 3 0)
            (2 1 1 3 3) (2 0 3 2 3) (2 2 2 2 3) (3 2 3 1 3))) ✠
[ligne deleted]
> (setf K (k-complex/i 5 I)) ✠
[K3 Chain-Complex]
> (setf eh (efhm K)) ✠
[K1235 Equivalence K3 <= K1208 => K1211]
```

```
> (seventh (homology-gen (k 1211) 3)) ✠
---------------------------------------------------------------------{CMBN 3}
<1 * <Con0 <Con0 <Con1 <Con0 <Con1 <Con1 Z-GNRT>>>>>>>
------------------------------------------------------------------------------
> (rg EH *) ✠
---------------------------------------------------------------------{CMBN 3}
<-1 * <Con0 <Con0 <Con0 <Con0 <Con0 <Con0 <Con0 <Con0 <Con0 ...
                  ... <Con0 <Con0 <Con0 ((2 1 2 3 2) (0 1 1 0 1))>>>>>>>>>>>>>
[... 8 lines deleted ...]
<1 * <Con0 <Con0 <Con1 <Con0 <Con1 <Con1 ((0 0 0 0 0) (0 0 0 0 0))>>>>>>>
------------------------------------------------------------------------------
> (lf EH *) ✠
---------------------------------------------------------------------{CMBN 3}
<-1 * ((2 1 2 3 2) (0 1 1 0 1))>
<1 * ((2 1 3 2 2) (0 1 0 1 1))>
<-1 * ((2 2 2 2 2) (0 0 1 1 1))>
------------------------------------------------------------------------------
```

Therefore a cycle representing our seventh 3-homology class is $-v^2wx^2y^3z^2\, dw.dx.dz \;+\; v^2wx^3y^2z^2\, dw.dy.dz \;-\; v^2w^2x^2y^2z^2\, dx.dy.dz$. Note the terrible cone towers which are involved. But all these calculations are almost instantaneous, and having tools doing them conveniently will soon become mandatory in modern homological algebra.

It is not obvious the last element actually is a homology generator, but we can at least verify it is a cycle!

```
> (? K *) ✠
---------------------------------------------------------------------{CMBN 2}
------------------------------------------------------------------------------
```

When the first argument is a chain-complex, here `K`, a symbol pointing to our Koszul complex $\mathrm{Ksz}_*(\overline{\mathfrak{R}}/\overline{I}')$, and the second argument is a combination, here '`*`' a symbol pointing to the last result returned, this combination must be an element of the chain-complex and the operator '`?`' computes the boundary of this combination in this chain-complex. Absence of terms between both horizontal dash lines means the result is null.

6.5 Case of a non-monomial ideal.

The work around the *monomial* ideal $\overline{I}'$ in the previous section was undertaken because we hope to be able to apply the BPL to obtain a version with effective homology $\mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I})_{EH}$ in the general case from $\mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I}')_{EH}$ now available. Thanks to a Grobner basis of $\overline{I}$, the $\mathfrak{k}$-vector spaces $\overline{\mathfrak{R}}/\overline{I}$ and $\overline{\mathfrak{R}}/\overline{I}'$ are canonically isomorphic, which implies the corresponding Koszul complexes are also isomorphic as graded $\mathfrak{k}$-vector spaces. Only the differentials are different, we are in situation to apply the BPL.

What about the nilpotency condition? A simple example explains better what happens than a generic description. Let us take $\overline{I} = <x - t^3, y - t^5> \subset \mathbb{Q}[x,y,t]$; the DegRevLex reduced Groebner basis for the order $x > y > t$ is $\overline{I} = <xt^2 - y,\ t^3 - x,\ x^2 - yt>$ and the associated monomial ideal is $\overline{I}' = <xt^2, t^3, x^2>$. Both quotients $\overline{\mathfrak{R}}/\overline{I}$ and $\overline{\mathfrak{R}}/\overline{I}'$ are isomorphic $\mathfrak{k}$-vector spaces with basis $\cup_{\alpha\in\mathbb{N}}\{y^\alpha, xy^\alpha, ty^\alpha, xty^\alpha, t^2y^\alpha\}$. A generator of a Koszul complex is a product of such a basis element and a combination of dx, dy and dt without any repetition. The differential is obtained by successively replacing the various *d?* by *?* with the right signs. If ever the resulting coefficient is not in our basis, two cases:

1. In the monomial case, the reduction modulo the ideal cancels the corresponding term.
2. In the initial non-monomial case, a reduction modulo the ideal in general generates other monomials.

For example $d_{\mathrm{Ksz}}(xt\,dt) = xt^2$ which is not in our basis, hence to be reduced; modulo $\overline{I}'$, the result is null; modulo $\overline{I}$, because of the generator $xt^2 - y$, the result is non null, it is y. *The main point is here*: because of the structure of the Groebner basis, the multigrading of the result is certainly *strictly less* than the multigrading of the initial monomial. In our small example, the multigrading of $xt\,dt$ is $[1,0,2]$ while the multigrading of y is $[0,1,0] < [1,0,2]$ for DegRevLex in $\mathbb{Q}[x,y,t]$.

Proposition 92 — *Let $\overline{I} \subset \overline{\mathfrak{R}}$ an ideal. Some Groebner monomial order is given for the multigrading. Cancelling the trailing terms of the corresponding reduced Groebner basis defined an "approximate" monomial ideal $\overline{I}'$, allowing us to identify* as multigraded $\mathfrak{k}$-vector spaces *the Koszul complexes* $\mathrm{Ksz}_*(\overline{\mathfrak{R}}/\overline{I})$ *and* $\mathrm{Ksz}_*(\overline{\mathfrak{R}}/\overline{I}')$. *Then the perturbation difference between both differentials* strictly decreases the multigrading. ■

Theorem 91 constructs an equivalence:

$$\mathrm{Ksz}_*(\overline{\mathfrak{R}}/\overline{I}') \lll \widehat{C}'_* \ggg EC'_*$$

with an effective chain-complex EC'_*. We would like to construct:

$$\mathrm{Ksz}_*(\overline{\mathfrak{R}}/\overline{I}) \lll \widehat{C}_* \ggg EC_*$$

As usual, applying the Easy Perturbation Lemma 49 between $\mathrm{Ksz}_*(\overline{\mathfrak{R}}/\overline{I}')$ and $\mathrm{Ksz}_*(\overline{\mathfrak{R}}/\overline{I})$ will produce the wished chain-complex $\widehat{C}_*$, the same graded vector space as $\widehat{C}'_*$ but with another differential. Then applying the serious Basic Perturbation Lemma 50 produces a new effective chain-complex EC_*, the same graded vector space as EC'_* with another differential.

The only critical point is the nilpotency hypothesis. The initial equivalence produced by Theorem 91 is entirely made of objects, differentials, morphisms, homotopy operators that are, thanks to the *multigrading shift* process, *multigraded.*

The initial perturbation between Koszul complexes on the contrary *strictly* decreases the multigrading. The easy perturbation lemma copies this perturbation into $\widehat{C}'_*$ using multigraded morphisms; therefore the differential perturbation to be applied to $\widehat{C}'_*$ to obtain $\widehat{C}_*$ also *strictly decreases the multigrading.* Now the composition homotopy-perturbation $h\widehat{\delta}$ which must be proved locally nilpotent is made of a multigraded map and another map which strictly decreases the multigrading; the composition also strictly decreases the multigrading.

A monomial order defines a well-founded order in the multigrading set; every strictly decreasing sequence goes to the minimal element, the multigrading of 0 often decided to be $-\infty$ and our composition $h\widehat{\delta}$ is nilpotent for any argument.

Theorem 93 — *An algorithm computes:*

$$\overline{I} \mapsto [\mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I}) \Lleftarrow \widehat{C}_* \Rrightarrow EC_*]$$

where $\overline{I}$ is an ideal of $\overline{\mathfrak{R}} = \mathfrak{k}[x_1, \ldots, x_m]$, and the result is an equivalence between the corresponding Koszul complex and an effective chain-complex of finite-dimensional $\mathfrak{k}$-vector spaces. ∎

6.6 Coming back to the local ring.

There remains to come back to our local ring $\mathfrak{R} = \mathfrak{k}[x_1, \ldots, x_m]_0$. The only difference between the elements of $\overline{\mathfrak{R}}$ and $\mathfrak{R}$ is that denominators are allowed for $\mathfrak{R}$, on condition such a denominator is non null at 0. A canonical inclusion is defined $\overline{\mathfrak{R}} \subset \mathfrak{R}$. The ring $\overline{\mathfrak{R}}$ is factorial and an element of $\mathfrak{R}$ can be written in a unique irreducible form $p/(1-m)$ with $p \in \overline{\mathfrak{R}}$ and $m \in \mathfrak{m}_0$, the maximal ideal of $\overline{\mathfrak{R}}$ at 0.

Theorem 94 — *Let I be an ideal of $\mathfrak{R}$ and $\overline{I} = I \cap \overline{\mathfrak{R}}$. The injection $\lambda : \overline{\mathfrak{R}} \hookrightarrow \mathfrak{R}$ induces an injection $\lambda : \mathrm{Ksz}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}}/\overline{I}) \hookrightarrow \mathrm{Ksz}_{\mathfrak{R}}(\mathfrak{R}/I)$ which in turn induces an* isomorphism*:*

$$\lambda : H_*(\mathrm{Ksz}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}}/\overline{I})) \xrightarrow{\cong} H_*(\mathrm{Ksz}_{\mathfrak{R}}(\mathfrak{R}/I)).$$

In short, the denominators do not play any role in the homological nature of these Koszul complexes.

Proof. Let us qualify as *polynomial* a chain element of the chain-complex $\mathrm{Ksz}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}}/\overline{I})$: all the coefficients are (equivalence classes of) *polynomials.* Every polynomial has a (total) degree and also an *order*, the smallest degree of a non-null monomial component, which definitions are extended to polynomial chains, without taking account of the "differential" terms in $\wedge V$. The $\mathfrak{k}$-vector space $H_*(\mathrm{Ksz}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}}/\overline{I}))$ has a finite dimension; choosing cycles representing some *generators* of this homology, the degree of every generator is $< k$ for some $k \in \mathbb{N}$. We will carefully examine which happens when objects are reduced modulo $\mathfrak{m}_0^k$.

The space of all cycles, *after reduction* modulo $\mathfrak{m}_0^k$, is also a finite dimensional vector space where the classes modulo $\mathfrak{m}_0^k$ of boundaries are a supplementary of the space generated by the chosen cycles representing the generators of homology: $Z' = H \oplus B'$ if H is the vector space generated by our (exact) representants, if Z' (resp. B') is the set of all cycles (resp. boundaries) *truncated* at degree k. In particular any cycle z of order $\geq k$ certainly is a boundary; in fact the homology class of z is obtained as follows: you truncate the cycle z at degree k, obtaining an element $z' \in Z'$ and the homology class "is" the H-component h of $z' = h + b'$. But if the cycle has an order $\geq k$, then $z' = 0$.

Let us take now a "local" cycle $z \in Z_*(\mathrm{Ksz}_{\mathfrak{R}}(\mathfrak{R}/I))$. Reducing to the same denominator the various components of z, this cycle can be written $z = \overline{z}/(1-m)$ with $\overline{z} \in Z_*(\mathrm{Ksz}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}}/\overline{I}))$ and $m \in \mathfrak{m}_0$. For $\overline{z} = (1-m)z$ again is a cycle: the differential is a *module* morphism. Now $\overline{z}/(1-m) = (\overline{z} + m\overline{z} + \cdots + m^{k-1}\overline{z}) + m^k\overline{z}/(1-m)$. Because of its order, the *numerator* $m^k\overline{z}$ is a boundary in the polynomial Koszul complex, which allows to express also the fraction $m^k\overline{z}/(1-m)$ as a boundary in the localized Koszul complex, again because the boundary operator is a *module* morphism. The sum of the other terms $\overline{z}+m\overline{z}+\cdots+m^{k-1}\overline{z}$ is polynomial, it is again a cycle and its homology class $\mathfrak{h}$ in the polynomial Koszul complex is defined; the previous study shows the homology class of z in the localized Koszul complex is $\lambda(\mathfrak{h})$ and λ at the homological level is surjective.

Let us take now $\overline{z} \in Z_*(\mathrm{Ksz}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}}/\overline{I}))$ and assume $\overline{z}$ is a boundary in $\mathrm{Ksz}_{\mathfrak{R}}(\mathfrak{R}/I)$, that is with the same calculation as before: $\overline{z} = d(c/(1-m)) = d(c + mc + \cdots + m^{k-1}c) + d(m^k c/(1-m)))$ with c a *polynomial* chain in $\mathrm{Ksz}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}}/\overline{I})$. So that in fact the last term $d(m^k c/(1-m))$ is a difference between polynomial chains and it is also polynomial; furthermore the computation of $d(d(m^k c/(1-m)))$ can be done as well in the localized Koszul complex; the result is null ($dd = 0$) and our pseudo-fraction $d(m^k c/(1-m))$ is also a cycle in the polynomial Koszul complex. Because of the order, this polynomial cycle is a boundary *in the polynomial Koszul complex* and finally the cycle $\overline{z}$ is a boundary in the polynomial Koszul complex. In other words the map λ at the homological level is injective. ∎

It is not very hard to transform this proof into a $(\overline{\mathfrak{R}}, \mathfrak{k}, \mathfrak{k})$-linear reduction $\mathrm{Ksz}_{\mathfrak{R}}(\mathfrak{R}/I) \Rrightarrow \mathrm{Ksz}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}}/\overline{I})$. But the most appropriate conclusion is the following: the homological problems for $\mathrm{Ksz}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}}/\overline{I}))$ and $\mathrm{Ksz}_{\mathfrak{R}}(\mathfrak{R}/I)$ are *constructively* equivalent. We have seen in the previous section how the homological problem for $\mathrm{Ksz}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}}/\overline{I})$ is solved thanks to two essential ingredients: Groebner basis and BPL.

Theorem 95 — *The homological problem of* $\mathrm{Ksz}_{\mathfrak{R}}(\mathfrak{R}/I)$ *is solved.* ∎

6.7 Effective homology ⇔ Effective resolution.

Let I be an ideal of our local ring $\mathfrak{R} = \mathfrak{k}[x_1, \ldots, x_m]_0$. We know how to compute the effective homology of $\mathrm{Ksz}(\mathfrak{R}/I)$. We intend now to use this information to obtain an *effective* $\mathfrak{R}$-resolution of $\mathfrak{R}/I$. If the effective homology is *minimal*, in a natural sense to make precise in due time, then the corresponding resolution is minimal too. Conversely, a (minimal) effective resolution naturally gives a (minimal) effective homology for the Koszul complex.

As before, it is better to work with $\overline{I} = I \cap \overline{\mathfrak{R}}$. Elementary arguments show an $\overline{\mathfrak{R}}$-resolution $\mathrm{Rsl}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}}/\overline{I})$ induces an $\mathfrak{R}$-resolution $\mathrm{Rsl}_{\mathfrak{R}}(\mathfrak{R}/I) := \mathrm{Rsl}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}}/\overline{I}) \otimes_{\overline{\mathfrak{R}}} \mathfrak{R}$. In particular the $\overline{\mathfrak{R}}$-module $\mathfrak{R}$ is *flat*.

The connection between effective homology of $\mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I})$ and effective resolution of $\overline{\mathfrak{R}}/\overline{I}$ is the Aramova-Herzog bicomplex [3].

Definition 96 — Let $\overline{I}$ be an ideal of $\overline{\mathfrak{R}}$. The *Aramova-Herzog bicomplex* $\mathrm{ArHr}(\overline{\mathfrak{R}}/\overline{I})$ of $\overline{\mathfrak{R}}/\overline{I}$ is $\mathrm{ArHr}(\overline{\mathfrak{R}}/\overline{I}) := \overline{\mathfrak{R}}/\overline{I} \otimes_{\mathfrak{k}} \wedge V \otimes_{\mathfrak{k}} \overline{\mathfrak{R}}$ provided with both differentials coming from both Koszul complexes present in its definition.

We recall V is the $\mathfrak{k}$-vector space $\mathfrak{m}_0/\mathfrak{m}_0^2$ provided with the canonical basis $(dx_1, \ldots, dx_m)$, the ideal $\mathfrak{m}_0$ being the maximal ideal at 0 of $\overline{\mathfrak{R}}$. We can see $\mathrm{ArHr}(\overline{\mathfrak{R}}/\overline{I}) := \overline{\mathfrak{R}}/\overline{I} \otimes \wedge V \otimes \overline{\mathfrak{R}} = \mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I}) \otimes \overline{\mathfrak{R}}$ and the *vertical* differential ∂'' of our bicomplex is $\partial'' := d_{\mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I})} \otimes \mathrm{id}_{\overline{\mathfrak{R}}}$. In the same way, appropriately swapping the factors $\wedge V$ and $\overline{\mathfrak{R}}$, we can interpret $\mathrm{ArHr}(\overline{\mathfrak{R}}/\overline{I}) := \overline{\mathfrak{R}}/\overline{I} \otimes \wedge V \otimes \overline{\mathfrak{R}} = \overline{\mathfrak{R}}/\overline{I} \otimes \mathrm{Ksz}(\overline{\mathfrak{R}})$ and the *horizontal* differential is $\partial' := \mathrm{id}_{\overline{\mathfrak{R}}/\overline{I}} \otimes d_{\mathrm{Ksz}(\overline{\mathfrak{R}})}$. See the following diagram where the ground ring $\overline{\mathfrak{R}}$ is split into its homogeneous components $\overline{\mathfrak{R}}_p$, this index p defining the horizontal grading, which implies the horizontal differential has degree $(0, +1)$. In the same way, the central factor $\wedge V$ is split into homogeneous components $\wedge^v V$, the vertical degree is $q = v + p$ but this time the vertical differential has degree $(-1, 0)$. The total degree therefore is v. The bicomplex is null outside the strip $0 \leq v \leq m$, that is, $q \in [p \mathinner{..} p + m]$.

$$
\begin{tikzcd}
{} \arrow[d, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & {} \arrow[d, dotted] & \\
\overline{\mathfrak{R}}/\overline{I} \otimes \wedge^3 \otimes \overline{\mathfrak{R}}_0 \arrow[r, "\partial'"] \arrow[d, "\partial''"'] & \overline{\mathfrak{R}}/\overline{I} \otimes \wedge^2 \otimes \overline{\mathfrak{R}}_1 \arrow[r, "\partial'"] \arrow[d, "\partial''"'] & \overline{\mathfrak{R}}/\overline{I} \otimes \wedge^1 \otimes \overline{\mathfrak{R}}_2 \arrow[r, "\partial'"] \arrow[d, "\partial''"'] & \overline{\mathfrak{R}}/\overline{I} \otimes \wedge^0 \otimes \overline{\mathfrak{R}}_3 \arrow[r] \arrow[d] & 0 \\
\overline{\mathfrak{R}}/\overline{I} \otimes \wedge^2 \otimes \overline{\mathfrak{R}}_0 \arrow[r, "\partial'"] \arrow[d, "\partial''"'] & \overline{\mathfrak{R}}/\overline{I} \otimes \wedge^1 \otimes \overline{\mathfrak{R}}_1 \arrow[r, "\partial'"] \arrow[d, "\partial''"'] & \overline{\mathfrak{R}}/\overline{I} \otimes \wedge^0 \otimes \overline{\mathfrak{R}}_2 \arrow[r] \arrow[d] & 0 & \\
\overline{\mathfrak{R}}/\overline{I} \otimes \wedge^1 \otimes \overline{\mathfrak{R}}_0 \arrow[r, "\partial'"] \arrow[d, "\partial''"'] & \overline{\mathfrak{R}}/\overline{I} \otimes \wedge^0 \otimes \overline{\mathfrak{R}}_1 \arrow[r] \arrow[d] & 0 & & \\
\overline{\mathfrak{R}}/\overline{I} \otimes \wedge^0 \otimes \overline{\mathfrak{R}}_0 \arrow[r] \arrow[d] & 0 & & & \\
0 & & & &
\end{tikzcd}
$$

If we see $\mathrm{ArHr}(\overline{\mathfrak{R}}/\overline{I}) = \overline{\mathfrak{R}}/\overline{I} \otimes \mathrm{Ksz}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}})$, using the fact the Koszul complex of the ground ring $\mathrm{Ksz}_{\overline{\mathfrak{R}}}(\overline{\mathfrak{R}})$ is acyclic (Theorem 67, or more precisely the variant for $\overline{\mathfrak{R}}$, easier), we will construct a reduction $\mathrm{ArHr}(\overline{\mathfrak{R}}/\overline{I}) \Rrightarrow \overline{\mathfrak{R}}/\overline{I}$. Considering now the symmetric factorization $\mathrm{ArHr}(\overline{\mathfrak{R}}/\overline{I}) = \mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I}) \otimes \overline{\mathfrak{R}}$, using the effective homology $\mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I}) \Lleftarrow\!\!\Rrightarrow H$, that is, an equivalence between the Koszul complex and some *effective* chain-complex H, we will construct an equivalence $\mathrm{ArHr}(\overline{\mathfrak{R}}/\overline{I}) \lll\!\!\Longleftrightarrow\!\!\ggg H \otimes \overline{\mathfrak{R}}$ with an appropriate differential for $H \otimes \overline{\mathfrak{R}}$ coming again from the BPL. Combining this reduction and this equivalence will produce an equivalence $\overline{\mathfrak{R}}/\overline{I} \lll\!\!\Longleftrightarrow\!\!\ggg H \otimes \overline{\mathfrak{R}}$ which is the looked-for resolution. If H is the minimal complex describing the homology of the Koszul complex, then the obtained resolution is minimal too. And the whole process can be reversed, starting from *effective* resolution*s*, going to *effective* homolog*ies* of the Koszul complex.

Let us recall the possible geometrical interpretation of the Koszul complex given Section 5.3. A natural analogous interpretation can be given here. The Koszul complex $\mathrm{Ksz}(M) = M \otimes_t \wedge V$ is the total space of a fibration $M \hookrightarrow M \otimes_t \wedge V \to \wedge V$. We can also consider the symmetric Koszul complex $\mathrm{Ksz}'(\mathfrak{R}) = \wedge V \,{}_t\!\otimes \mathfrak{R}$ with an analogous fibration. Combining both fibrations gives the diagram:

$$\begin{tikzcd}
 & & M \arrow[d, hook] \\
 & M \otimes_t \wedge V \,{}_t\!\otimes \mathfrak{R} \arrow[r] \arrow[d] & M \otimes_t \wedge V \arrow[d] \\
\mathfrak{R} \arrow[r, hook] & \wedge V \,{}_t\!\otimes \mathfrak{R} \arrow[r] & \wedge V
\end{tikzcd}$$

where the Aramova-Herzog bicomplex $M \otimes_t \wedge V \,{}_t\!\otimes \mathfrak{R}$ is the *pullback* of the vertical fibration by the horizontal map $\wedge V \,{}_t\!\otimes \mathfrak{R} \longrightarrow \wedge V$; but the original space of this map is contractible (Theorem 67), so that it has the homotopy type of a point, and the pullback, up to homotopy, is nothing but the base fiber of the vertical fibration, that is, the module M. It is this homotopy equivalence which is systematically exploited by Aramova and Herzog. Note the vertical arrow between $M \otimes_t \wedge V$ and $\wedge V$ is not actually defined and there is only some "analogy" with the projection of a topological fibration.

First reduction.

The chain-complex $\mathrm{Ksz}(\overline{\mathfrak{R}})$ is acyclic. More precisely, every "horizontal" *sub*complex $(\oplus_{v+p=q} \wedge^v V \otimes \overline{\mathfrak{R}}_p, \partial')$ at ordinate q is acyclic, except for $q = 0$ where $\wedge^0 V \otimes \overline{\mathfrak{R}}_0 = \mathfrak{k}$. Appling the functor $\overline{\mathfrak{R}}/\overline{I} \otimes <?>$ gives a reduction $(\mathrm{ArHr}(\overline{\mathfrak{R}}/\overline{I}), \partial') \Rrightarrow \overline{\mathfrak{R}}/\overline{I}$, where, quite important, the vertical differential ∂'' has temporarily been cancelled. Reinstalling the vertical differential is BPL's job. Verifying the nilpotency hypothesis is the following game: you start from $\mathrm{ArHr}_{p,q}(\overline{\mathfrak{R}}/\overline{I})$, a homotopy operator expressing ∂' is contractible leads you to $\mathrm{ArHr}_{p-1,q}(\overline{\mathfrak{R}}/\overline{I})$, the perturbation ∂'' goes to $\mathrm{ArHr}_{p-1,q-1}(\overline{\mathfrak{R}}/\overline{I})$, the next homotopy operator goes to $\mathrm{ArHr}_{p-2,q-1}(\overline{\mathfrak{R}}/\overline{I})$ and so on. Finally you get out from the diagram at $\mathrm{ArHr}_{0,q-p}(\overline{\mathfrak{R}}/\overline{I})$ after having run p steps of a stairs leftdownward.

Second equivalence.

The analogous work for the second interpretation of the Aramova-Herzog bicomplex works as follows. We now consider $\mathrm{ArHr}(\overline{\mathfrak{R}}/\overline{I}) = \mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I}) \otimes \overline{\mathfrak{R}}$. Theorem 93 constructs an equivalence $\mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I}) \lll\!\!\Longleftrightarrow\!\!\ggg H$ where H is a chain-complex of finite type, called H for it describes the "abstract" homology of the Koszul complex. This equivalence can be applied to every vertical of the Aramova-Herzog bicomplex, which produces an equivalence $(\mathrm{ArHr}(\overline{\mathfrak{R}}/\overline{I}), \partial'') \lll\!\!\Longrightarrow\!\!\ggg H \otimes \overline{\mathfrak{R}}$ with $d_{H \otimes \overline{\mathfrak{R}}} = d_H \otimes \mathrm{id}_{\overline{\mathfrak{R}}}$.

There remains to reinstall the horizontal differential ∂', again under the responsability of BPL. The nilpotency check runs the same stairs as before, but in the *reverse direction*, and this time we *do not reach* any void part of the bicomplex. But the vertical homotopy operator comes from Theorem 93 and the details of the proof show this homotopy operator *does not* increase the Groebner multidegree: in the monomial case, this operator is multigraded and respects the multigrading; in the general case, Shih's magic formula $h' = h\psi = h\sum_{i=0}^{\infty}(-1)^i(\widehat{\delta}h)^i$, see page 49, gives the result because of Proposition 92.

We were speaking here of the multigrading of $\mathrm{ArHr}(\overline{\mathfrak{R}}/\overline{I}) = \mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I}) \otimes \overline{\mathfrak{R}}$ deduced from the left hand factor $\mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I})$, neglecting the right hand factor $\overline{\mathfrak{R}}$. If we consider the relevant perturbation ∂', every term of $\partial'(\kappa \otimes v \otimes \rho)$ is obtained by replacing some dx_i in $v \in \wedge V$ by the corresponding x_i to be installed as a multiplier in $\rho \in \overline{\mathfrak{R}}$. This *strictly* decreases the "left hand" multigrading of $\mathrm{ArHr}(\overline{\mathfrak{R}}/\overline{I}) = \mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I}) \otimes \overline{\mathfrak{R}}$. The nilpotency condition is satisfied.

Applying BPL is allowed, which gives an equivalence:

$$(\mathrm{ArHr}(\overline{\mathfrak{R}}/\overline{I}), \partial' \oplus \partial'') \lll\!\!\Longrightarrow\!\!\ggg (H \otimes \overline{\mathfrak{R}}, d')$$

with a *new* differential $d' \neq d_h \otimes \mathrm{id}_{\overline{\mathfrak{R}}}$ except in trivial cases. Combining the first reduction and the second equivalence gives the next theorem.

Theorem 97 — *The Aramova-Herzog bicomplex ArHr*$(\overline{\mathfrak{R}}/\overline{I})$ *produces an equivalence:*

$$\overline{\mathfrak{R}}/\overline{I} \lll\!\!\Longrightarrow\!\!\ggg (H \otimes \overline{\mathfrak{R}}, d')$$

■

The left hand term of this equivalence is without any differential, and more exactly is a chain-complex concentrated in differential degree 0. Our equivalence is nothing but a resolution $(H \otimes \overline{\mathfrak{R}}, d')$ for $\overline{\mathfrak{R}}/\overline{I}$. The component H is a free (!) $\mathfrak{k}$-vector space and the tensor product $H \otimes \overline{\mathfrak{R}}$ therefore is a *free* $\overline{\mathfrak{R}}$-module. The possibly sophisticated differential d', sophisticated but *automatically* produced by BPL, describes the main part of the resolution.

Elementary Gauss reductions allow one to *reduce* an H with a non-null differential over another one with null differential. In this case the equivalence $\mathrm{Ksz}(\overline{\mathfrak{R}}/\overline{I}) \lll\!\!\Longrightarrow\!\!\ggg H$ is the *minimal* effective homology of the Koszul complex, unique up to some appropriate… equivalence. Then the $\overline{\mathfrak{R}}$-ranks of the components of the obtained resolution $(H \otimes \overline{\mathfrak{R}}, d')$ are minimal as well: a minimal effective homology produces a minimal resolution.

6.8 Examples.

6.8.1 The minimal non-trivial example.

Let $\mathfrak{R} = \mathfrak{k}[x]$ (one variable) and $M = \mathfrak{R}/ < x^2 >$. And let us assume we do not know (!) the minimal resolution. Here the ideal is monomial and the steps 1 and 3 of our algorithm are void. The effective homology of the Koszul complex:

$$\mathrm{Ksz}(M) = [\cdots \leftarrow 0 \leftarrow M \leftarrow M.dx \leftarrow 0 \leftarrow \cdots]$$

is made of the chain-complex:

$$H = [\cdots \leftarrow 0 \leftarrow \mathfrak{k}_0 \overset{0}{\leftarrow} \mathfrak{k}_1 \leftarrow 0 \leftarrow \cdots]$$

(where $\mathfrak{k}_0$ and $\mathfrak{k}_1$ are copies of the ground field $\mathfrak{k}$ with respective homological degrees 0 and 1) *and* of the maps $\rho = (f, g, h)$ with:

1. $f : M \to \mathfrak{k}_0$ is defined by $f(1) = 1_0, f(x) = 0$.
2. $f : M.dx \to \mathfrak{k}_1$ is defined by $f(1.dx) = 0, f(x.dx) = 1_1$.
3. $g : \mathfrak{k}_0 \to M$ is defined by $g(1_0) = 1$.
4. $g : \mathfrak{k}_1 \to M.dx$ is defined by $g(1_1) = x.dx$.
5. $h : M \to M.dx$ is defined by $h(1) = 0, h(x) = 1.dx$.

We must guess the right differential on $\mathrm{Rsl}(M) = (H \otimes_{\mathfrak{k}} \mathfrak{R}, d =?)$. The only non-trivial differential $d_{\mathrm{Rsl}(M)}(1_1 \otimes 1_{\mathfrak{R}})$ comes from a unique non-null term in the series (Σ), following the path:

$$1_1 \otimes 1_{\mathfrak{R}} \overset{g\otimes \mathrm{id}_{\mathfrak{R}}}{\longmapsto} x \otimes dx \otimes 1_{\mathfrak{R}} \overset{\partial'}{\mapsto} x \otimes 1 \otimes x \overset{-h\otimes \mathrm{id}_{\mathfrak{R}}}{\longmapsto} -1 \otimes dx \otimes x \overset{\partial'}{\longmapsto} -1 \otimes 1 \otimes x^2 \overset{f\otimes \mathrm{id}_{\mathfrak{R}}}{\longmapsto} -1_0 \otimes x^2$$

and, surprise, we find the resolution $1_1 \otimes 1_{\mathfrak{R}} \mapsto -1_0 \otimes x^2$. You find it is a little complicated for a so trivial particular case? The point is the following: this example in a sense is *complete*, the most general case is not harder, you have here all the ingredients of the general solution, nothing more is necessary.

6.8.2 First Aramova-Herzog example.

In the paper [3], Aramova and Herzog consider the toy example of the ideal $I = < x_1x_3, x_1x_4, x_2x_3, x_2x_4 >$ in $\mathfrak{R} = \mathfrak{k}[x_1, x_2, x_3, x_4]$. The ideal is monomial and again, steps 1 and 3 of our algorithm are void. The Betti numbers of $\mathrm{Ksz}(\mathfrak{R}/I)$ are $(1, 4, 4, 1)$ and the effective homology of $\mathrm{Ksz}(\mathfrak{R}/I)$ is a diagram:

$$\rho = \boxed{h \circlearrowright \mathrm{Ksz}(\mathfrak{R}/I) \underset{f}{\overset{g}{\leftrightarrows}} H}$$

where H is the chain-complex with null differentials:

$$\cdots \longleftarrow \mathfrak{k} \xleftarrow{0} \mathfrak{k}^4 \xleftarrow{0} \mathfrak{k}^4 \xleftarrow{0} \mathfrak{k} \longleftarrow \cdots$$

The arrows f and g are chain-complex morphisms satisfying $fg = \mathrm{id}_H$, the self-arrow h is a homotopy between gf and $\mathrm{id}_{\mathrm{Ksz}(\mathfrak{R}/I)}$, that is, $\mathrm{id}_{\mathrm{Ksz}(\mathfrak{R}/I)} = gf + dh + hd$, and finally, the composite maps fh, hg and h^2 are null. These maps smartly express the big chain-complex $\mathrm{Ksz}(\mathfrak{R}/I)$ as the direct sum of the small one H, in this case with trivial differentials, and an acyclic one ($\ker f$) *with* an *explicit* contraction h. Our Kenzo program [19] computes this effective homology in a negligible time with respect to input-output. In particular the map g defines representants for the alleged homology classes, the map f is a projection which in particular sends cycles to their homology classes, and h is the main component of a *constructive* proof of these claims.

The minimal resolution of $\mathfrak{R}/I$ is $\mathrm{Rsl}(\mathfrak{R}/I) = H \otimes \mathfrak{R}$ where a non-trivial differential must be installed. Let us apply our formula to the unique generator $\mathfrak{h}_{3,1} \otimes 1_{\mathfrak{R}}$ of $H_3 \otimes \mathfrak{R}$. Kenzo chooses $g(\mathfrak{h}_{3,1}) = x_2\, dx_1.dx_3.dx_4 - x_1\, dx_2.dx_3.dx_4$ and:

$$\begin{array}{rl} \partial'(g \otimes 1_{\mathfrak{R}})(\mathfrak{h}_{3,1} \otimes 1_{\mathfrak{R}}) = & x_2\, dx_3.dx_4 \otimes x_1 \\ & -x_1\, dx_3.dx_4 \otimes x_2 \\ & +(-x_2\, dx_1.dx_4 + x_1\, dx_2.dx_4) \otimes x_3 \\ & +(x_2\, dx_1.dx_3 - x_1\, dx_2.dx_3) \otimes x_4 \end{array}$$

Kenzo is a little luckier than Aramova and Herzog, for he had chosen:

$$\begin{array}{rcl} g(\mathfrak{h}_{2,1}) & = & -x_2\, dx_1.dx_3 + x_1\, dx_2.dx_3 \\ g(\mathfrak{h}_{2,2}) & = & -x_1\, dx_3.dx_4 \\ g(\mathfrak{h}_{2,3}) & = & -x_2\, dx_1.dx_4 + x_1\, dx_2.dx_4 \\ g(\mathfrak{h}_{2,4}) & = & -x_2\, dx_3.dx_4 \end{array}$$

which is enough to imply:

$$d(\mathfrak{h}_{3,1}) = -\mathfrak{h}_{2,1} \otimes x_4 + \mathfrak{h}_{2,2} \otimes x_2 + \mathfrak{h}_{2,3} \otimes x_3 - \mathfrak{h}_{2,4} \otimes x_1$$

that is, except for legal minor differences, directly the same result as Aramova and Herzog.

Let us now *force* Kenzo to choose Aramova and Herzog's representants for the homology classes of H_2. This amounts to replacing the component g in degree 2 by another one $g' = g + d\alpha$ for α a map $\alpha : H_2 \to \mathrm{Ksz}_3(\mathfrak{R}/I)$ chosen to give the new representants. The cycle $-x_2\, dx_1.dx_i + x_1\, dx_2.dx_i$ ($i = 3$ or 4) is homologous to the cycle $-x_i\, dx_1.dx_2$ (sign error in [3]) thanks to the boundary preimage $dx_1.dx_2.dx_i$. So that we transform Kenzo's choices to Aramova and Herzog's choices by taking $\alpha(\mathfrak{h}_{2,1}) = -dx_1.dx_2.dx_3$, $\alpha(\mathfrak{h}_{2,3}) = -dx_1.dx_2.dx_4$ and $\alpha(\mathfrak{h}_{2,i}) = 0$ for $i = 2$ or 4.

The component f of the reduction does not change, but the homotopy h_2 must be replaced by $h'_2 = h_2(\mathrm{id} - d\alpha f_2)$. Repeating the same computation, taking account of $g_3 = g'_3$, now the homotopy term $(h'_2 \otimes \mathrm{id}_{\mathfrak{R}})\partial'(g_3 \otimes \mathrm{id}_{\mathfrak{R}})(\mathfrak{h}_{3,1}) =$

$dx_1.dx_2.dx_4 \otimes x_3 - dx_1.dx_2.dx_3 \otimes dx_4$ *is not null*, so that we must continue the expansion of the series (Σ). We find:

$$\begin{array}{rcl} -\partial'(h'_2 \otimes \mathrm{id}_{\mathfrak{R}})\partial'(g \otimes \mathrm{id}_{\mathfrak{R}})(\mathfrak{h}_{3,1}) & = & -dx_2.dx_4 \otimes x_1x_3 + dx_1.dx_4 \otimes x_2x_3 \\ & & +dx_2.dx_3 \otimes x_1x_4 - dx_1.dx_3 \otimes x_2x_4 \end{array}$$

but applying f or h' to the left hand factors of the tensor products this time gives 0 and the final result is the same: Aramova-Herzog's conclusion is so justified; the *possible* pure nature of the looked-for resolution, known in advance after examining the Koszul cycles, may also be used to cancel the examination of the critical homotopy operator, but we will see our method can be applied in much more general situations, even in a non-homogeneous situation. In more complicated situations, the result could have been different: "the" minimal resolution is unique only up to chain-complex isomorphism and this set of isomorphisms is very large. In this particular case, many triangular perturbations can for example be applied to the simple expression found for $d(\mathfrak{h}_{3,1})$ without changing its intrinsic nature, and in parallel the same for "the" effective homology of the Koszul complex.

Another comment is also necessary. After all, any (correct) choice for the representants $g(\mathfrak{h}_{2,i})$ is possible, so that why it would not be possible to prefer Kenzo's choices to the initial unfortunate choices by Aramova and Herzog? The point is the following: a resolution is not only made of isomorphism classes of the boundary maps, you must make these maps fit to each other in such a way there is *equality* between appropriate kernels and images. So that when you change the cycles representing the homology classes during the computation of the component d_3 of the resolution for example, then the computation of d_2 could also be modified.

6.8.3 Second Aramova-Herzog example.

On one hand it is significantly simpler than the first one: the concerned module is a $\mathfrak{k}$-vector space of finite dimension 3, so that any computation is elementary. On another hand it is a little harder: the interesting differential to be constructed is quadratic. Note in particular it was not obvious in the previous example to obtain the effective homology: the concerned module was a $\mathfrak{k}$-vector space of infinite dimension, but the standard methods of effective homology know how to overcome such a problem; in fact they were invented exactly to *overcome* such a problem, see [52].

The underlying ground ring now is $\mathfrak{R} = \mathfrak{k}[x_1, x_2]$ and we consider the module $M = <x_1, x_2>/<x_1^2, x_2^2>$. The module M is a $\mathfrak{k}$-vector space of dimension 3. The Koszul complex is of dimension 3 in degrees 0 and 2, of dimension 6 in degree 1.

The simplest form of the effective homology is well described by this figure.

	$\mathrm{Ksz}_0(M) = \mathfrak{k}^3$	$\mathrm{Ksz}_1(M) = \mathfrak{k}^6$	$\mathrm{Ksz}_2(M) = \mathfrak{k}^3$
R_1		$x_1\,dx_2$	$-x_1\,dx_1.dx_2$ $x_2\,dx_1.dx_2$
R_2	x_1x_2	$x_1x_2\,dx_1$ $x_1x_2\,dx_2$	
R_3	x_1 x_2	$x_2\,dx_1 - x_1\,dx_2$ $x_1\,dx_1$ $x_2\,dx_2$	$x_1x_2\,dx_1.dx_2$

Each column corresponds to a component of the Koszul complex and the (almost) canonical basis is shared in boundary preimages, cycles homologous to zero, and homology classes, each homology class being represented by a cycle not at all homologous to zero. The effective homology:

$$\rho = \boxed{h \circlearrowleft \mathrm{Ksz}(M) \underset{f}{\overset{g}{\leftrightarrows}} H}$$

is read on the figure as follows. The map g consists in representing the homology classes by the cycles listed on the bottom row R_3. The map f is the inverse projection which forgets the basis vectors of the rows R_1 and R_2. The differentials and the homotopy operator h are simultaneously represented by bidirectional arrows. The chosen supplementary of the homology groups – in fact of the representing cycles – are shared in two components (R_1 and R_2) isomorphic through the differential in the decreasing direction, through the homotopy operator in the increasing direction. This diagram expresses in a very detailed way the Betti numers are $(2, 3, 1)$.

The chain-complex H is $[0 \leftarrow \mathfrak{k}^2 \leftarrow \mathfrak{k}^3 \leftarrow \mathfrak{k} \leftarrow 0]$ with null differentials. We have to install the right differential on $H \otimes \mathfrak{R}$. With the same notations as in the previous section, the differential d_2 of the minimal resolution is obtained by a unique non-null term of the series (Σ) following the path:

$$\begin{array}{rcl} & & \mathfrak{h}_{2,1} \\ (g_2 \otimes \mathrm{id}_{\mathfrak{R}}) & \mapsto & x_1x_2\,dx_1.dx_2 \\ \partial' & \mapsto & x_1x_2\,dx_2 \otimes x_1 - x_1x_2\,dx_1 \otimes x_2 \\ -(h_1 \otimes \mathrm{id}_{\mathfrak{R}}) & \mapsto & -x_2\,dx_1.dx_2 \otimes x_1 - x_1\,dx_1.dx_2 \otimes x_2 \\ \partial' & \mapsto & -x_2\,dx_2 \otimes x_1^2 + (x_2\,dx_1 - x_1\,dx_2) \otimes x_1x_2 + x_1\,dx_1 \otimes x_2^2 \\ (f_1 \otimes \mathrm{id}_{\mathfrak{R}}) & \mapsto & -\mathfrak{h}_{1,3} \otimes x_1^2 - \mathfrak{h}_{1,1} \otimes x_1x_2 + \mathfrak{h}_{1,2} \otimes x_2^2, \end{array}$$

that is, the same result as in [3], except innocent sign changes and permutations. All the other terms produced by the series (Σ) are null.

The "path" described above makes also obvious the nilpotency argument which guarantees the convergence of the series (Σ): in $M \otimes \wedge V \otimes \mathfrak{R}$, the central term $\wedge V$

"inhales" the monomials from the left hand factor M and partly "exhales" them to the right hand side after some processing, giving back also something on the left hand side but with a strictly inferior degree. After a finite number of steps, certainly nothing anymore on the left hand side. This is particularly clear in the homogeneous case, a little more difficult but interesting in the general case: the Groebner monomial orders again play an important role here.

You see in fact the nature of this example is *essentially* the same as for our initial "minimal non-trivial" example.

6.8.4 The favourite Kreuzer-Robbiano example.

Martin Kreuzer and Lorenzo Robbiano use a little more complicated toy example in their book [34, Chapter 4], in fact close to the first Aramova-Herzog example. Again the ring $\mathfrak{R} = \mathfrak{k}[x_1, x_2, x_3, x_4]$ but the ideal is nomore monomial: $I = \langle x_2^3 - x_1^2x_3,\ x_1x_3^2 - x_2^2x_4, x_3^3 - x_2x_4^2, x_2x_3 - x_1x4\rangle$. It is a Groebner basis for DegRevLex, so that step 1 of the algorithm is void, but the ideal is nomore monomial and step 3 is not. Keeping the leading terms, we consider the close ideal $I' = \langle x_2^3, x_1x_3^2, x_3^3, x_2x_3\rangle$. It is a monomial ideal and the effective homology of the Koszul complex $\mathrm{Ksz}(\mathfrak{R}/I')$ is easily computed; the Betti numbers are $(1, 4, 4, 1)$ and Kenzo gives for example as a generator of the 3-homology the cycle $-x_3^2\, dx_1.dx_2.dx_3$. Applying the homological perturbation lemma to take account of the difference between I and I' gives the effective homology of $\mathrm{Ksz}(\mathfrak{R}/I)$; the new Betti numbers are certainly bounded by the previous ones, but in this simple case, they are the same. The generator of the homology in dimension 3 is now $-x_3^2\, dx_1.dx_2.dx_3 + x_2x_4\, dx_1.dx_2.dx_4 - x_1x_3\, dx_1.dx_3.dx_4 + x_2^2\, dx_2.dx_3.dx_4$. There remains to play the same game with the components f, g and h of the effective homology, and also with the differential ∂' of the Aramova-Herzog bicomplex, exactly the same game as before, nothing more, to obtain the minimal resolution:

$$0 \longleftarrow \mathfrak{R} \xleftarrow{d_1} \mathfrak{R}^4 \xleftarrow{d_2} \mathfrak{R}^4 \xleftarrow{d_3} \mathfrak{R} \longleftarrow 0$$

with the matrices:

$$d_1 = \left[\ x_1^2x_3 - x_2^3, -x_1x_3^2 + x_2^2x_4, x_2x_4^2 - x_3^3, -x_1x_4 + x_2x_3\ \right]$$

$$d_2 = \begin{bmatrix} 0 & -x_3 & -x_4 & 0 \\ -x_3 & -x_1 & -x_2 & x_4 \\ x_1 & 0 & 0 & -x_2 \\ x_2x_4 & -x_2^2 & -x_1x_3 & -x_3^2 \end{bmatrix} \qquad d_3 = \begin{bmatrix} -x_2 \\ -x_4 \\ x_3 \\ -x_1 \end{bmatrix}$$

Another toy example.

Let us finally consider now the non-homogeneous ideal:

$$I = \langle t^5 - x, t^3y - x^2, t^2y^2 - xz, t^3z - y^2, t^2x - y, tx^2 - z, x^3 - ty^2, y^3 - x^2z, xy - tz\rangle$$

This ideal seems more complicated than the previous one, but in a sense in fact it is not. This ideal is obtained by applying the DegRevLex Groebner process to $I = \langle x - t^5, y - t^7, z - t^{11} \rangle$ and the simple arithmetic nature of the toric generators allows us to expect a simple minimal resolution. But the program ignores this expression of I and it is interesting to observe the result of its study: the minimal resolution is in principle a machine to analyze the *deep* structure of an ideal or module. Macaulay2's `resolution` gives for $\mathfrak{R}/I$ a resolution with Betti numbers $(1, 7, 11, 6, 1)$ which is not minimal[23]. On the contrary, Singulars `mres` computes the minimal resolution, necessarily equivalent to ours; but to our knowledge, Singular does not give any information about the connection between the homology of the Koszul complex and this minimal resolution, in particular between the *effective* character of the homology of the Koszul complex and the *effective* character of the obtained resolution. No indication in [25] about these subjects.

The approximate monomial module $\mathfrak{R}/I'$ has Betti numbers $(1, 9, 15, 8, 1)$. Applying the homological perturbation lemma between $\mathrm{Ksz}(\mathfrak{R}/I')$ and $\mathrm{Ksz}(\mathfrak{R}/I)$ gives the effective homology of the last one. The Betti numbers are, surprise, $(1, 3, 3, 1)$. For example a generator for the 3-homology is $-x^2\, dt.dx.dy + tx\, dt.dx.dz - t^4 dt.dy.dz + dx.dy.dz$. The same process as before using the Aramova-Herzog bicomplex now describes a possible minimal resolution. The differentials can be:

$$d_1 = \left[-t^2x + y, -tx^2 + z, -t^5 + x\right]$$

$$d_2 = \begin{bmatrix} 0 & t^5 - x & tx^2 - z \\ t^5 - x & 0 & -tx^2 + y \\ -tx^2 + z & -t^2x + y & 0 \end{bmatrix}$$

$$d_3 = \begin{bmatrix} -t^2x + y \\ tx^2 - z \\ -t^5 + x \end{bmatrix}$$

With respect to the series (Σ), each term of degree k in the previous matrices comes from a term of the series with $i = k - 1$. Here all the terms of the series are null for $i \geq 5$: in fact the degree corresponds to the number of applications of ∂'.

7 Simplicial sets.

7.1 Introduction.

To illustrate in Section 2.2.2 how the *chain-complexes* can be used, the notion of *simplicial complex* was defined. The general organization of traditional algebraic topology is roughly as explained in the diagram:

Topology $\rightarrow$ Combinatorial Topology $\rightarrow$ Chain-Complexes $\rightarrow$ Homology Groups

[23] But the writer of the part of this text is not at all a Macaulay2 expert; using the rich set of Macaulay2 procedures, it is certainly possible to compute the minimal resolution.

Constructive algebraic topology must improve this framework. On one hand, *locally effective* objects are systematically used to *implement*, theoretically or concretely, the infinite objects which are quickly unavoidable. On the other hand, a systematic connection with *effective* objects must be maintained during the construction steps, currently the only method allowing one to easily produce *algorithms* computing the traditional invariants: homology groups, homotopy groups, Postnikov (pseudo-)invariants...

The notion of simplicial complex is the most elementary method to settle a connection between common "general" topology and homological algebra. The "sensible" spaces can be triangulated, at least up to homotopy, and instead of using the notion of topological space, too "abstract", only the spaces having the homotopy type of a CW-complex (see [35]) are considered, and all these spaces in turn have the homotopy type of a simplicial complex. So that a lazy algebraic topologist can decide every space is a simplicial complex.

But many common *constructions* in topology are difficult to make explicit in the framework of simplicial complexes. It soon became clear in the forties the tricky and elegant notion of simplicial *set* is much better. It is the subject of this section. The reference [40] certainly remains the basic reference in this subject; it is a book of Mathematics' Gold Age, when a reasonable detail level was naturally required, and in this respect, this book is perfect; in particular many explicit formulas, quite useful if you want to *constructively* work, can be found only in this book. A unique flaw: no concrete examples; the present section must be understood just as a reading help to Peter May's book, providing the "obvious" examples that are necessary to understand the exact motivation of this subtle notion of simplicial set and the related definitions; these examples are obvious, except for the beginner. Combining both, you should be quickly able to work yourself with this wonderful tool.

7.2 The category Δ.

Some strongly structured sets of indices are necessary to define the notion of *simplicial object*; they are conveniently organized as the category Δ. An object of Δ is a set $\underline{\mathbf{m}}$, namely the set of integers $\underline{\mathbf{m}} = \{0, 1, \ldots, m-1, m\}$; this set is canonically *ordered* with the usual order between integers.

A Δ-morphism $\alpha : \underline{\mathbf{m}} \to \underline{\mathbf{n}}$ is an *increasing* map. Equal values are permitted; for example a Δ-morphism $\alpha : \underline{\mathbf{2}} \to \underline{\mathbf{3}}$ could be defined by $\alpha(0) = \alpha(1) = 1$ and $\alpha(2) = 3$. The set of Δ-morphisms from $\underline{\mathbf{m}}$ to $\underline{\mathbf{n}}$ is denoted by $\Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}})$; the subset of injective (resp. surjective) morphisms is denoted by $\Delta^{\mathrm{inj}}(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ (resp. $\Delta^{\mathrm{srj}}(\underline{\mathbf{m}}, \underline{\mathbf{n}})$).

Some *elementary* morphisms are important, namely the simplest non-surjective and non-injective morphisms. For geometric reasons explained later, the first ones are the *face morphisms*, the second ones are the *degeneracy morphisms*.

Definition 98 — The *face morphism* $\partial_i^m : \underline{\mathbf{m-1}} \to \underline{\mathbf{m}}$ is defined for $m \geq 1$ and

$0 \leq i \leq m$ by:

$$\begin{array}{rll} \partial_i^m(j) = & j & \text{if } j < i, \\ \partial_i^m(j) = & j+1 & \text{if } j \geq i. \end{array}$$

The face morphism ∂_i^m is the unique injective morphism from $\underline{\mathbf{m-1}}$ to $\underline{\mathbf{m}}$ such that the integer i is not in the image. The face morphisms generate the injective morphisms, in fact in a unique way if a growth condition is required.

Proposition 99 — *Any injective Δ-morphism $\alpha \in \Delta^{\mathrm{inj}}(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ has a unique expression:*

$$\alpha = \partial_{i_n}^n \circ \ldots \circ \partial_{i_{m+1}}^{m+1}$$

satisfying the relation $i_n > i_{n-1} > \ldots > i_{m+1}$.

Proof. The index set $\{i_{m+1}, \ldots, i_n\}$ is exactly the difference set $\underline{\mathbf{n}} - \alpha(\underline{\mathbf{m}})$, that is, the set of the integers where surjectivity fails. ∎

Frequently the upper index m of ∂_i^m is omitted because clearly deduced from the context. For example the unique injective morphism $\alpha : \underline{\mathbf{2}} \to \underline{\mathbf{5}}$ the image of which is $\{0, 2, 4\}$ can be written $\alpha = \partial_5 \partial_3 \partial_1$.

If two face morphisms are composed in the wrong order, they can be exchanged: $\partial_i \circ \partial_j = \partial_{j+1} \circ \partial_i$ if $j \geq i$. Iterating this process allows you to quickly compute for example $\partial_0 \partial_2 \partial_4 \partial_6 = \partial_9 \partial_6 \partial_3 \partial_0$.

Definition 100 — The *degeneracy morphism* $\eta_i^m : \underline{\mathbf{m+1}} \to \underline{\mathbf{m}}$ is defined for $m \geq 0$ and $0 \leq i \leq m$ by:

$$\begin{array}{rll} \eta_i^m(j) = & j & \text{if } j \leq i, \\ \eta_i^m(j) = & j-1 & \text{if } j > i. \end{array}$$

The degeneracy morphism η_i^m is the unique surjective morphism from $\underline{\mathbf{m+1}}$ to $\underline{\mathbf{m}}$ such that the integer i has two pre-images. The degeneracy morphisms generate the surjective morphisms, in fact in a unique way if a growth condition is required.

Proposition 101 — *Any surjective Δ-morphism $\alpha \in \Delta^{\mathrm{srj}}(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ has a unique expression:*

$$\alpha = \eta_{i_n}^n \circ \ldots \circ \eta_{i_{m-1}}^{m-1}$$

satisfying the relation $i_n < i_{n+1} < \ldots < i_{m-1}$.

Proof. The index set $\{i_n, \ldots, i_{m-1}\}$ is exactly the set of integers j such that $\alpha(j) = \alpha(j+1)$, that is, the integers where injectivity fails. ∎

Frequently the upper index m of η_i^m is omitted because clearly deduced from the context. For example the unique surjective morphism $\alpha : \underline{\mathbf{5}} \to \underline{\mathbf{2}}$ such that $\alpha(0) = \alpha(1)$ and $\alpha(2) = \alpha(3) = \alpha(4)$ can be expressed $\alpha = \eta_0 \eta_2 \eta_3$.

If two face morphisms are composed in the wrong order, they can be exchanged: $\eta_i \circ \eta_j = \eta_j \circ \eta_{i+1}$ if $i \geq j$. Iterating this process allows you to quickly compute for example $\eta_3 \eta_3 \eta_2 \eta_2 = \eta_2 \eta_3 \eta_5 \eta_6$.

Proposition 102 — *Any Δ-morphism α can be Δ-decomposed in a unique way:*

$$\alpha = \beta \circ \gamma$$

with β injective and γ surjective.

Proof. The intermediate Δ-object $\underline{\mathbf{k}}$ necessarily satisfies $k+1 = \mathrm{Card}(\mathbf{im}(\alpha))$. The growth condition then gives a unique choice for β and γ. ■

Corollary 103 — *Any Δ-morphism $\alpha : \underline{\mathbf{m}} \to \underline{\mathbf{n}}$ has a unique expression:*

$$\alpha = \partial_{i_n} \circ \ldots \circ \partial_{i_{k+1}} \circ \eta_{j_k} \circ \ldots \circ \eta_{j_{m-1}}$$

satisfying the conditions $i_n > \ldots > i_{k+1}$ and $j_k < \ldots < j_{m-1}$. ■

Finally if face and degeneracy morphisms are composed in the wrong order, they can be exchanged:

$$\begin{array}{rcll} \eta_i \circ \partial_j & = & \mathrm{id} & \text{if } j = i \text{ or } j = i+1; \\ & = & \partial_{j-1} \circ \eta_i & \text{if } j \geq i+2; \\ & = & \partial_j \circ \eta_{i-1} & \text{if } j < i. \end{array}$$

All these commuting relations can be used to convert an arbitrary composition of faces and degeneracies into the canonical expression:

$$\alpha = \eta_9\partial_6\eta_3\partial_7\eta_9\partial_8\eta_6\partial_2\eta_4\partial_9 = \partial_7\partial_6\partial_2\eta_2\eta_4\eta_6.$$

This relation means the image of α does not contain the integers 2, 6 and 7, and the relations $\alpha(2) = \alpha(3)$, $\alpha(4) = \alpha(5)$ and $\alpha(6) = \alpha(7)$ are satisfied.

Corollary 104 — *A* contravariant *functor $X : \Delta \to$* CAT *is nothing but a collection $\{X_m\}_{m\in\mathbb{N}}$ of objects of the target category* CAT*, and collections of* CAT*-morphisms $\{X(\partial_i^m) : X_m \to X_{m-1}\}_{m\geq 1\,,\,0\leq i\leq m}$ and $\{X(\eta_i^m) : X_m \to X_{m+1}\}_{m\geq 0\,,\,0\leq i\leq m}$ satisfying the commuting relations:*

$$\begin{array}{rcll} X(\partial_i)\circ X(\partial_j) & = & X(\partial_j)\circ X(\partial_{i+1}) & \text{if } i \geq j, \\ X(\eta_i)\circ X(\eta_j) & = & X(\eta_{j+1})\circ X(\eta_i) & \text{if } j \geq i, \\ X(\partial_i)\circ X(\eta_j) & = & \mathrm{id} & \text{if } i = j, j+1, \\ X(\partial_i)\circ X(\eta_j) & = & X(\eta_{j-1})\circ X(\partial_i) & \text{if } j > i, \\ X(\partial_i)\circ X(\eta_j) & = & X(\eta_j)\circ X(\partial_{i-1}) & \text{if } i > j+1. \end{array}$$

In the five last relations, the upper indices have been omitted. Such a contravariant functor is a *simplicial object* in the category CAT. If α is an arbitrary Δ-morphism, it is then sufficient to express α as a composition of face and degeneracy morphisms; the image $X(\alpha)$ is necessarily the composition of the images of the corresponding $X(\partial_i)$'s and $X(\eta_i)$'s; the above relations assure the definition is coherent.

7.3 Terminology and notations.

Definition 105 — A *simplicial set* is a simplicial object in the category of sets.

A simplicial set X is given by a collection of sets $\{X(\underline{\mathbf{m}})\}_{m\in\mathbb{N}}$ and collections of maps $\{X_\alpha\}$, the index α running the Δ-morphisms; the usual coherence properties must be satisfied. As explained at the end of the previous section, it is sufficient to define the $X(\partial_i^m)$'s and the $X(\eta_i^m)$'s with the corresponding commuting relations.

The set $X(\underline{\mathbf{m}})$ is usually denoted by X_m and is called the set of m-simplices of X; such a simplex has the *dimension m*. To be a little more precise, these simplices are sometimes called *abstract* simplices, to avoid possible confusions with the *geometric* simplices defined a little later. An (abstract) m-simplex is only *one* element of X_m.

If $\alpha \in \Delta(\underline{\mathbf{n}}, \underline{\mathbf{m}})$, the corresponding morphism $X(\alpha) : X_m \to X_n$ is most often simply denoted by $\alpha^* : X_m \to X_n$ or still more simply $\alpha : X_m \to X_n$. In particular the faces and degeneracy operators are maps $\partial_i : X_m \to X_{m-1}$ and $\eta_i : X_m \to X_{m+1}$. If σ is an m-simplex, the (abstract) simplex $\partial_i\sigma$ is its i-th face, and the simplex $\eta_i\sigma$ is its i-th degeneracy; we will see the last one is "particularly" abstract.

7.4 The structure of simplex sets.

Definition 106 — An m-simplex σ of the simplicial set X is *degenerate* if there exists an integer $n < m$, an n-simplex $\tau \in X_n$ and a Δ-morphism $\alpha \in \Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ such that $\sigma = \alpha(\tau)$. The set of non-degenerate simplices of dimension m in X is denoted by X_m^{ND}.

Decomposing the morphism $\alpha = \beta \circ \gamma$ with γ surjective, we see that $\sigma = \gamma(\beta(\tau))$, with the dimension of $\beta(\tau)$ less or equal to n; so that in the definition of degenaracy, the connecting Δ-morphism α can be required to be surjective. The relation $\sigma = \alpha(\tau)$ with α surjective is shortly expressed by saying the m-simplex σ *comes from* the n-simplex τ.

Eilenberg's lemma explains each degenerate simplex comes from a canonical non-degenerate one.

Lemma 107 — **(Eilenberg's lemma)** *If X is a simplicial set and σ is an m-simplex of X, there exists a unique triple $T_\sigma = (n, \tau, \alpha)$ satisfying the following conditions:*

1. *The first component n is a natural number $n \leq m$;*
2. *The second component τ is a non-degenerate n-simplex $\tau \in X_n^{ND}$;*
3. *The third component α is a Δ-morphism $\tau \in \Delta^{\mathrm{srj}}(\underline{\mathbf{m}}, \underline{\mathbf{n}})$;*
4. *The relation $\sigma = \alpha(\tau)$ is satisfied.*

Definition 108 — This triple T_σ is called the *Eilenberg triple* of σ.

Proof. Let $\mathcal{T}$ be the set of triples $T = (n, \tau, \alpha)$ such that $n \leq m$, $\tau \in X_n$ and $\alpha \in \Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ satisfy $\sigma = \alpha(\tau)$. The set $\mathcal{T}$ certainly contains the triple (m, σ, id) and therefore is non empty. Let (n_0, τ_0, α_0) be an element of $\mathcal{T}$ where the first component, the integer n_0, is minimal. We claim (n_0, τ_0, α_0) is the Eilenberg triple.

Certainly $n_0 \leq m$. The n_0-simplex τ_0 is non-degenerate; otherwise $\tau_0 = \beta(\tau_1)$ with the dimension n_1 of τ_1 less than n_0, but then $(n_1, \tau_1, \beta\alpha_0)$ would be a triple with $n_1 < n_0$. Finally α_0 is surjective, otherwise $\alpha_0 = \beta\gamma$ with $\gamma \in \Delta^{\mathrm{srj}}(m, n_1)$ and $n_1 < n_0$; but again the triple $(n_1, \beta(\tau_0), \gamma)$ would be a triple denying the required property of n_0. The existence of an Eilenberg triple is proved and uniqueness remains to be proved.

Let (n_1, τ_1, α_1) be another Eilenberg triple. The morphisms α_0 and α_1 are surjective and respective sections $\beta_0 \in \Delta^{\mathrm{inj}}(\underline{\mathbf{n_0}}, \underline{\mathbf{m}})$ and $\beta_1 \in \Delta^{\mathrm{inj}}(\underline{\mathbf{n_1}}, \underline{\mathbf{m}})$ can be constructed: $\alpha_0\beta_0 = \mathrm{id}$ and $\alpha_1\beta_1 = \mathrm{id}$. Then $\tau_0 = (\alpha_0\beta_0)(\tau_0) = \beta_0(\alpha_0(\tau_0)) = \beta_0(\sigma) = \beta_0(\alpha_1(\tau_1)) = (\alpha_1\beta_0)(\tau_1)$; but τ_0 is non-degenerate, so that $n_1 = \dim(\tau_1) \geq n_0 = \dim(\tau_0)$; the analogous relation holds when τ_0 and τ_1 are exchanged, so that $n_1 \leq n_0$ and the equality $n_0 = n_1$ is proved.

The relation $\tau_0 = \beta_0(\alpha_1(\tau_1))$ with τ_0 non-degenerate implies $\alpha_1\beta_0 = \mathrm{id}$, otherwise $\alpha_1\beta_0 = \gamma\delta$ with $\delta \in \Delta^{\mathrm{srj}}(\underline{\mathbf{n_1}}, \underline{\mathbf{n_2}})$ and $n_2 < n_1 = n_0$, but this implies τ_0 comes from $\gamma(\tau_1)$ of dimension n_2 again contradicting the non-degeneracy property of τ_0; therefore $\alpha_1\beta_0 = \mathrm{id}$ but this equality implies $\tau_0 = \tau_1$.

If $\alpha_0 \neq \alpha_1$, let i be an integer such that $\alpha_0(i) = j \neq \alpha_1(i)$; then the section β_0 can be chosen with $\beta_0(j) = i$; but this implies $(\alpha_1\beta_0)(j) \neq j$, so that the relation $\alpha_1\beta_0 = \mathrm{id}$ would not hold. The last required equality $\alpha_0 = \alpha_1$ is also proved. ■

Each simplex comes from a unique non-degenerate simplex, and conversely, for any non-degenerate m-simplex $\sigma \in X_m^{ND}$, the collection $\{\alpha(\sigma)\,;\, \alpha \in \Delta^{\mathrm{srj}}(\underline{\mathbf{n}}, \underline{\mathbf{m}})\,;\, n \geq m\}$ is a perfect description of all simplices coming from σ, that is, of all degenerate simplices *above* σ. This is also expressed in the following formula, describing the structure of the simplex set of any simplicial set X:

$$\coprod_{m \in \mathbb{N}} X_m = \coprod_{m \in \mathbb{N}} \; \coprod_{\sigma \in X_m^{ND}} \; \coprod_{n \geq m} \Delta^{\mathrm{srj}}(\underline{n}, \underline{m})(\sigma).$$

7.5 Examples.

7.5.1 Discrete simplicial sets.

Definition 109 — A simplicial set X is *discrete* if $X_m = X_0$ for every $m \geq 1$, and if for every $\alpha \in \Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}})$, the induced map $\alpha^* : X_n \to X_m$ is the identity.

The reason of this definition is that the *realization* (see Section 7.6) of such a simplicial set is the discrete point set X_0; the Eilenberg triple of any simplex $\sigma \in X_m = X_0$ is $(0, \sigma, \alpha)$ where the map α is the unique element of $\Delta(\underline{\mathbf{m}}, \underline{\mathbf{0}})$.

7.5.2 The simplicial complexes.

A *simplicial complex* $K = (V, S)$ is a pair where V is the *vertex set* (an arbitrary set, finite or not), and $S \subset \mathcal{P}_F(V)$ is a set of finite sets of vertices satisfying the properties:

1. For any $v \in V$, the one element subset $\{v\}$ of V is an element of S;
2. For any $\tau \subset \sigma \in S$, then $\tau \in S$.

The *simplex* $\sigma \in S$ *spans* its elements. If $S = \mathcal{P}_F(V)$, then K is the *simplex* freely generated by V, or more simply the simplex spanned by V.

The terminology is a little incoherent because a simplicial *set* is an object more sophisticated than a simplicial *complex*, but this terminology is so well established that it is probably too late to modify it.

The simplicial complex $K = (V, S)$ is *ordered* if the vertex set V is provided with a *total* order[24]. Then a simplicial set again denoted by K is canonically associated; the simplex set K_m is the set of *increasing* maps $\sigma : \underline{\mathbf{m}} \to K$ such that the image of $\underline{\mathbf{m}}$ is an element of S; note that such a map σ is not necessarily injective. If α is a Δ-morphism $\alpha \in \Delta(\underline{\mathbf{n}}, \underline{\mathbf{m}})$ and σ is an m-simplex $\sigma \in K_m$, then $\alpha(\sigma)$ is naturally defined as $\alpha(\sigma) = \sigma \circ \alpha$. A simplex $\sigma \in K_m$ is non-degenerate if and only if $\sigma \in \Delta^{\text{inj}}(\underline{\mathbf{m}}, V)$; if $\sigma \in K_m = \Delta(\underline{\mathbf{m}}, V)$, the Eilenberg triple (n, τ, α) satisfies $\sigma = \tau \circ \alpha$ with α surjective and τ injective.

This in particular works for $K = (\underline{\mathbf{d}}, \mathcal{P}(\underline{\mathbf{d}}))$ the simplex freely generated by $\underline{\mathbf{d}}$ provided with the canonical vertex order. We obtain in this way the canonical structure of simplicial set for the *standard d-simplex* Δ^d.

7.5.3 The spheres.

Let d be a natural number. The simplest simplicial version $S = S^d$ of the d-sphere is defined as follows: the set of m-simplices S_m is $S_m = \{*_m\} \coprod \Delta^{\text{srj}}(\underline{\mathbf{m}}, \underline{\mathbf{d}})$; if $\alpha \in \Delta(\underline{\mathbf{n}}, \underline{\mathbf{m}})$ and σ is an m-simplex $\sigma \in S_m$, then $\alpha(\sigma)$ depends on the nature of σ:

1. If $\sigma = *_m$, then $\alpha(\sigma) = *_n$;
2. Otherwise $\sigma \in \Delta^{\text{srj}}(\underline{\mathbf{m}}, \underline{\mathbf{d}})$ and if $\sigma \circ \alpha$ is surjective, then $\alpha(\sigma) = \sigma \circ \alpha$, else $\alpha(\sigma) = *_n$ (the emergency solution when the natural solution does not work).

This is nothing but the canonical quotient $S^d = \Delta^d / \partial\Delta^d$, at least if $d > 0$; more generally the notion of simplicial subset is naturally defined and a quotient then appears. In the case of the construction of $S^d = \Delta^d / \partial\Delta^d$, the subcomplex $\partial\Delta^d$ is made of the simplices $\alpha \in \Delta(\underline{\mathbf{m}}, \underline{\mathbf{d}})$ that are not surjective.

The Eilenberg triple of $*_m$ is $(0, *_0, \alpha)$ where *alpha* is the unique element of $\Delta(\underline{\mathbf{m}}, \underline{\mathbf{0}})$. The Eilenberg triple of $\sigma \in \Delta^{\text{srj}}(\underline{\mathbf{m}}, \underline{\mathbf{d}}) \subset S_m$ is (d, id, σ). There are only two non-degenerate simplices, namely $*_0 \in S_0$ and $\text{id}(\underline{\mathbf{d}}) \in S_d$, even if $d = 0$.

[24] Other situations where the order is not total are also interesting but will be considered later.

7.5.4 Classifying spaces of discrete groups.

Let G be a (discrete) group. Then a simplicial version of its classifying space BG can be given. The set of m-simplices BG_m is the set of "m-bars" $\sigma = [g_1|\ldots|g_m]$ where every g_i is an element of G. It is simpler in this situation to define the structure morphisms only for the face and degeneracy operators:

1. $\partial_0[g_1|\ldots|g_m] = [g_2|\ldots|g_m]$;
2. $\partial_m[g_1|\ldots|g_m] = [g_1|\ldots|g_{m-1}]$;
3. $\partial_i[g_1|\ldots|g_m] = [\ldots|g_{i-1}|g_i g_{i+1}|g_{i+2}|\ldots]$ if $0 < i < m$;
4. $\eta_i[g_1|\ldots|g_m] = [\ldots|g_i|e_G|g_{i+1}|\ldots]$, where e_G is the unit element of G.

In particular $BG_0 = \{[\,]\}$ has only one element.

The m-simplex $[g_1|\ldots|g_m]$ is degenerate if and only if one of the G-components is the unit element.

The various commuting relations must be verified; this works but does not give obvious indications on the very nature of this construction; in fact there is a more conceptual description. Let us define the simplicial set EG by $EG_m = \mathrm{SET}(\underline{\mathbf{m}}, G)$, that is, the maps of $\underline{\mathbf{m}}$ to G without taking account of the ordered structure of $\underline{\mathbf{m}}$ (the group G is not ordered); if $\alpha \in \Delta(\underline{\mathbf{n}}, \underline{\mathbf{m}})$ there is a canonical way to define $\alpha : EG_m \to EG_n$; it would be more or less coherent to write $EG = G^{\Delta}$.

There is a canonical left action of the group G on EG, and BG is the natural quotient of EG by this action. A simplex $\sigma \in EG_m$ is nothing but a $(m+1)$-tuple $(g_0, \ldots, g_m)$ and the action of g gives the simplex $(gg_0, \ldots, gg_m)$. If two simplices are G-equivalent, the products $g_{i-1}^{-1} g_i$ are the same; the quotient BG-simplex $[g_1, \ldots, g_m]$ denotes the equivalence class of all the EG-simplices $(g, gg_1, gg_1g_2, \ldots)$, which can be imagined as a simplex where the *edge* between the vertices $i-1$ and i $(i > 0)$ is labeled by g_i to be considered as a (right) operator between the adjacent vertices. Then the boundary and degeneracy operators are clearly explained and it is even not necessary to prove the commuting relations, they can be deduced of the coherent structure of EG.

7.5.5 The Eilenberg-MacLane spaces.

The previous example constructs an *Eilenberg-MacLane* space, that is, a space with only one non-zero homotopy group. The *realization* process (see later) applied to the simplicial set BG produces a model for $K(G, 1)$: all the homotopy groups are null except π_1 canonically isomorphic to G. The construction can be generalized to construct $K(\pi, d)$, $d > 1$, when π is an *abelian* group. This requires the simplicial definition of homology groups, explained in another lecture series. See also [40, Chapter V] where these questions are carefuly detailed.

Let π be a fixed abelian group, and d a natural number. The simplicial set $E(\pi, d)$ is defined as follows. The set of m-simplices $E(\pi, d)_m$, shortly denoted by E_m, is $E_m = C^d(\Delta^m, \pi)$, the group of *normalized* d-cochains on the standard m-simplex with values in π. Such a cochain σ is nothing but a map $\sigma : \Delta^m_d \to \pi$, defined on the (degenerate or not) d-simplices of Δ^m, null for the degenerate simplices. If α is a Δ-morphism $\alpha : \underline{\mathbf{n}} \to \underline{\mathbf{m}}$, this map defines a simplicial map $\alpha_* : \Delta^n \to \Delta^m$ which in turns defines a pullback map $\alpha^* : C^d(\Delta^m, \pi) \to C^d(\Delta^n, \pi)$ between m-simplices and n-simplices of E_m.

The simplicial set $E(\pi, d)$ so defined contains the simplicial subset $K(\pi, d)$, made only of the *cocycles*, those cochains the coboundary of which ($d : C^d(\Delta^m, \pi) \to C^{d+1}(\Delta^m, \pi)$ is null. In fact $E(\pi, d)$ is a *simplicial group*, that is, a simplicial object in the category of groups, and $K(\pi, d)$ is a simplicial subgroup. The quotient simplicial group $E(\pi, d)/K(\pi, d)$ is canonically isomorphic to $K(\pi, d+1)$ and this structure defines the Eilenberg-MacLane fibration:

$$K(\pi, d) \hookrightarrow E(\pi, d) \to K(\pi, d+1)$$

See later the section about *simplicial fibrations* for some details.

7.5.6 Simplicial loop spaces.

Let X be a simplicial set. We can construct a new simplicial set $DT(X)$ (the acronym DT meaning Dold-Thom) from X, where $DT(X)_m$ is the free $\mathbb{Z}$-module generated by the m-simplices X_m; the operators of $DT(X)$ are also "generated" by the operators of X. This is a simplicial version of the Dold-Thom construction, producing a new simplicial set $DT(X)$, the homotopy groups of which being the homology groups of the initial X. The simplicial set $DT(X)$ is also of *simplicial group*; its simplex *sets* are nothing but the chain groups at the origin of the simplicial homology of X, but in $DT(X)$, each simplicial "chain" of X is *one* (abstract) simplex. See [40, Section 22].

The same construction can be undertaken, but instead of using the abelian group generated by the simplex sets X_m, we could consider the free *non-commutative* group generated by X_m. This also works, but then the obtained space is a simplicial model for the *James construction* of $\Omega\Sigma X$, the loop space of the (reduced) suspension of X. See [12] for the James construction in general and [16] for the simplicial case.

It is even possible to construct the "pure" loop space ΩX, without any suspension. This is due to Daniel Kan [32] and works as follows. It is necessary to assume X is reduced, that is with only one vertex: the cardinality of X_0 is 1. Let X_m^* the set of all m-simplices, except those that are 0-degenerate: $X_m^* = X_m - \eta_0(X_{m-1})$; this makes sense for $m \geq 1$. Then let GX_m be the free *non-commutative* group generated by X_{m+1}^*; to avoid possible confusions, if $\sigma \in X_{m+1}^*$, let us denote by $\tau(\sigma)$ the corresponding *generator* of GX_m. The simplicial object GX to be defined is a simplicial *group*, so that it is sufficient to define face and degeneracy operators for the generators:

$$\begin{array}{rcll} \partial_i \tau(\sigma) & = & \tau(\partial_{i+1}\sigma), & \text{if } 1 \leq i \leq m; \\ \partial_0 \tau(\sigma) & = & \tau(\partial_1\sigma)\tau(\partial_0\sigma)^{-1}; & \\ \eta_i \tau(\sigma) & = & \tau(\eta_{i+1}\sigma), & \text{if } 0 \leq i \leq m. \end{array}$$

These definitions are coherent, and the simplicial set GX so obtained is a simplicial version of the loop space construction. See [40, Chapter VI] for details and related questions, mainly the *twisted Eilenberg-Zilber Theorem*, at the origin of the general solution described in [57, 52] for the computability problem in algebraic topology.

7.5.7 The singular simplicial set.

Let X be an arbitrary topological space. Then the *singular simplicial set* associated with X is constructed as follows. The set of m-simplices SX_m is made of the continuous maps $\sigma : \Delta^m \to X$; *one* (abstract) simplex is *one* continuous map but no topology is installed on SX_m; in particular when SX will be *realized* in the following section, the *discrete* topology must be used. The source of the abstract m-simplex σ is the geometric m-simplex $\Delta^m \subset \mathbb{R}^{\underline{m}}$ provided with the traditional topology. If $\alpha \in \Delta(\underline{\mathbf{n}}, \underline{\mathbf{m}})$ is a Δ-morphism, this α defines a natural continuous map $\alpha_* : \Delta^n \to \Delta^m$ between geometric simplices, and this allows us to naturally define $\alpha^*(\sigma) = \sigma \circ \alpha_*$. An enormous simplicial set is so defined if X is an arbitrary topological space; it is at the origin of the *singular homology* theory.

7.6 Realization.

If $K = (V, S)$ is a simplicial set, the realization $|K|$ is a subset of $\mathbb{R}^{(V)}$, the $\mathbb{R}$-vector space generated by the vertices $v \in V$; a point $x \in \mathbb{R}^{(V)}$ is a function $x : V \to \mathbb{R}$ almost everywhere null, that is, the set of v's where x is non-null is finite. Such a function can also be denoted by $x = \{x_v\}_{v \in V}$, the set of indexed values, or also the linear notations $x = \sum x_v.e_v$ or $x = \sum x_v.v$ can be used. Then $|K|$ is the set of x's in $\mathbb{R}^{(V)}$ satisfying the following conditions:

1. For every $v \in V$, the inequality $0 \leq x_v \leq 1$ holds;
2. The relation $\sum_{v \in V} x_v = 1$ is satisfied;
3. The set $\{v \in V \;\; \mathbf{st} \;\; x_v \neq 0\}$ is a simplex $\sigma \in S$.

The right topology to install on $|K|$ is induced by all the finite dimensional spaces $\mathbb{R}^\sigma$ for $\sigma \in S$. In this way the realization $|K|$ is a CW-complex. In particular, if Δ^m is the simplex freely generated by $\underline{\mathbf{m}}$, the realization is the standard geometric m-simplex again denoted by Δ^m, provided with its ordinary topology. In general the topology of $|K|$ is induced by its (geometric) simplices.

If $\alpha : \underline{\mathbf{m}} \to \underline{\mathbf{n}}$ is a Δ-morphism, then α defines a covariant induced map $\alpha_* : \Delta^m \to \Delta^n$ (between the "simplicial" simplices or the geometric realizations, as you like) and for any simplicial set X a contravariant induced map $\alpha^* : X_n \to X_m$.From now on, unless otherwise stated, Δ^m denotes the *geometric* standard simplex, that is, the convex hull of the canonical basis of $\mathbb{R}^{\underline{m}}$.

If X is a simplicial set, the (*expensive*) realization $|X|$ of X is:

$$|X| = \coprod_{m \in \mathbb{N}} X_m \times \Delta^m \;/\; \approx .$$

Each component of the coproduct is the product of the discrete set of m-simplices by the geometric m-simplex; in other words, each abstract simplex σ in X_m gives birth to a geometric simplex $\{\sigma\} \times \Delta^m$, and they are attached to each other following the instructions of the equivalence relation $\approx$, to be defined. Let $\alpha \in \Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ be some Δ-morphism, and let σ be an n-simplex $\sigma \in X_n$ and $t \in \Delta^m \subset \mathbb{R}^{\underline{m}}$. Then the pairs $(\alpha^*\sigma, t)$ and $(\sigma, \alpha_* t)$ are declared equivalent.

It is not obvious to understand what is the topological space so obtained. A description a little more explicit but also a little more complicated explains more satisfactorily what should be understood.

The *cheap* realization $\|X\|$ of the simplicial set X is:

$$\|X\| = \coprod_{m \in \mathbb{N}} X_m^{ND} \times \Delta^m \ / \ \approx$$

where the equivalence relation $\approx$ is defined as follows. Let σ be a non-degenerate m-simplex and i an integer $0 \leq i \leq m$; let also $t \in \Delta^{m-1}$; the abstract $(m-1)$-simplex $\partial_i^*\sigma$ has a well defined Eilenberg triple (n, τ, α); then we decide to declare equivalent the pairs $(\sigma, \partial_{i*}(t)) \approx (\tau, \alpha_*(t))$.

For example let $S = S^d$ be the claimed simplicial version of the d-sphere described in Section 7.5.3. This simplical set S has only two non-degenerate simplices, one in dimension 0, the other one in dimension d. The cheap realization needs a point Δ^0 and a geometric d-simplex Δ^d corresponding to the abstract simplex $\mathrm{id} \in \Delta(\underline{\mathbf{d}}, \underline{\mathbf{d}})$; then if $t \in \Delta^{d-1}$ and $0 \leq i \leq d$, the equivalence relation asks for the Eilenberg triple of $\partial_i(\mathrm{id}) = *_{d-1}$ which is $(0, *_0, \eta)$, the map η being the unique element of $\Delta(\underline{\mathbf{d-1}}, \underline{\mathbf{0}})$. Finally the initial pair $(\mathrm{id}, \partial_{i*}t)$ in the realization process must be identified with the pair $(*_0, \Delta^0)$; in other words $\|S\| = \Delta^d / \partial\Delta^d$, homeomorphic to the unit d-ball with the boundary collapsed to one point.

Proposition 110 — *Both realizations, the expensive one and the cheap one, of a simplicial set X are canonically homeomorphic.*

Proof. The homeomorphism $f : |X| \rightarrow \|X\|$ to be constructed maps the equivalence class of the pair $(\sigma, t) \in X_m \times \Delta^m$ to the (equivalence class of the) pair $(\tau, \alpha_*(t)) \in X_n \times \Delta^n$ if the Eilenberg triple of σ is (n, τ, α). The inverse homeomorphism g is induced by the canonical inclusion $\coprod X_m^{ND} \times \Delta^m \hookrightarrow \coprod X_m \times \Delta^m$. These maps must be proved coherent with the defining equivalence relations and inverse to each other.

If $\alpha = \beta\gamma$ is a Δ-morphism expressed as the composition of two other Δ-morphisms, then an equivalence $(\sigma, \beta_*\gamma_* t) \approx (\gamma^*\beta^*\sigma, t)$ can be considered as a consequence of the relations $(\sigma, \beta_*\gamma_* t) \approx (\beta^*\sigma, \gamma_* t)$ and $(\beta^*\sigma, \gamma_* t) \approx (\gamma^*\beta^*\sigma, t)$, so that it is sufficient to prove the coherence of the definition of f with respect to the *elementary* Δ-operators, that is, the face and degeneracy operators.

Let us assume the Eilenberg triple of $\sigma \in X_m$ is (n, τ, α), so that $f(\sigma, t) = (\tau, \alpha_* t)$. We must in particular prove that $f(\eta_i^*\sigma, t)$ and $f(\sigma, \eta_{i*}t)$ are coherently defined. The second image is the equivalence class of $(\tau, \alpha_*\eta_{i*}t)$; the Eilenberg triple of $\eta_i^*\sigma$ is $(n,\tau,\alpha\eta_i)$ so that the first image is the equivalence class of $(\tau,(\alpha\eta_i)_*t)$ and both image representants are even equal.

Let us do now the analogous work with the face operator ∂_i instead of the degeneracy operator η_i. Two cases must be considered. If ever the composition $\alpha\partial_i \in \Delta(\underline{\mathbf{m-1}},\underline{\mathbf{n}})$ is surjective, the proof is the same. The interesting case happens if $\alpha\partial_i$ is not surjective; but its image then forgets exactly one element j $(0 \leq j \leq n)$ and there exists a unique surjection $\beta \in \Delta(\underline{\mathbf{m-1}},\underline{\mathbf{n-1}})$ such that $\alpha\partial_i = \partial_j\beta$. The abstract simplex $\partial_j^*\tau$ gives an Eilenberg triple (n',τ',α') and the unique possible Eilenberg triple for $\partial_i^*\sigma$ is $(n',\tau',\beta\alpha')$. Then, on one hand, the f-image of $(\sigma,\partial_{i*}t)$ is $(\tau,\alpha_*\partial_{i*}t) = (\tau,\partial_{j*}\beta_*t)$; on the other hand the f-image of $(\partial_i^*\sigma,t)$ is $(\tau',\alpha_*\beta_*t)$; but according to the definition of the equivalence relation $\approx$ for $\|X\|$, both f-images are equivalent. The coherence of f is proved.

Let $\sigma \in X_m^{ND}$, $0 \leq i \leq m$, $t \in \Delta^{m-1}$ and (n,τ,α) (the Eilenberg triple of $\partial_i^*\sigma$) be the ingredients in the definition of the equivalence relation for $\|X\|$; the pairs $(\sigma,\partial_{i*}t)$ and (τ,α_*t) are declared equivalent in $\|X\|$; the map g is induced by the canonical inclusion of coproducts, so that we must prove the same pairs are also equivalent in $|X|$. But this is a transitive consequence of $(\sigma,\partial_{i*}t) \approx (\partial_i^*\sigma,t) = (\alpha^*\tau,t) \approx (\tau,\alpha_*t)$. We see here we had only described the binary relations *generating* the equivalence relation $\approx$; the defining relation is not necessarily stable under transitivity. The coherence of g is proved.

The relation $fg = \mathrm{id}$ is obvious. The other relation $gf = \mathrm{id}$ is a consequence of the equivalence in $|X|$ of $(\sigma,t) \approx (\tau,\alpha_*t)$ if the Eilenberg triple of σ is (n,τ,α).

■

7.6.1 Examples.

Let us consider the construction of the classifying space of the group $G = \mathbb{Z}_2$ described in Section 7.5.4. The universal "total space" EG has for every $m \in \mathbb{N}$ exactly two non-degenerate m-simplices $(0,1,0,1,\ldots)$ and $(1,0,1,0,\ldots)$. The only non degenerate faces are the 0-face and the m-face. For example the faces of $(0,1,0,1)$ are $(1,0,1) \in EG_2^{ND}$, $(0,0,1) = \eta_0(0,1)$, $(0,1,1) = \eta_1(0,1)$ and $(0,1,0) \in EG_2^{ND}$. Each non-degenerate m-simplex is attached to the $(m-1)$-skeleton of EG like each hemisphere of S^m is attached to the equator S^{m-1} and EG is nothing but the infinite sphere S^∞. The details are not so easy; the key point consists in proving the geometric m-simplex corresponding for example to $\sigma = (0,1,0,1,\ldots)$ with a few identification relations on the boundary, following the instructions read from the various iterated faces of σ, is again homeomorphic to the m-ball, its boundary to the $(m-1)$-sphere; the simplest case is $\Delta^2/\partial_1\Delta^2 \cong D^2$, for $\partial_1\Delta^2 = \Delta^1$ is contractible, and this can be extended to the higher dimensions.

The classifying space BG is the quotient space of EG by the canonical action of $\mathbb{Z}_2$, that is, the quotient space of S^∞ by the corresponding action; so that BG is homeomorphic to the infinite real projective space $P^\infty\mathbb{R}$; the m-skeleton (throw away all the non-degenerate simplices of dimension $> m$ and also *their* degeneracies) is a combinatorial description of $P^m\mathbb{R}$. If $\sigma_m = [1|1|\ldots|1|1]$ denotes the unique non-degenerate simplex of BG; then $\partial_0 \sigma_m = \sigma_{m-1}$, $\partial_1 \sigma_m = \eta_0 \sigma_{m-2}$, $\ldots$, $\partial_{m-1} \sigma_m = \eta_{m-2} \sigma_{m-2}$ and $\partial_m \sigma_m = \sigma_{m-1}$.

Let us also consider the case of the singular simplicial set of a topological space X (see Section 7.5.7). There is a canonical continuous map $f : |SX| \to X$ defined as follows; if (σ, t) represents an element of $|SX|$, this means the (abstract) simplex σ is a continuous map $\sigma : \Delta^m \to X$, but t is an element of the geometric simplex Δ^m, so that it is tempting to define $f(\sigma, t) = \sigma(t)$; it is easy to verify this definition is coherent with the equivalence relation defining $|SX|$. This map is always a weak homotopy equivalence, and is an ordinary homotopy equivalence if and only if X has the homotopy type of a CW-complex.

7.6.2 Simplicial maps.

A natural notion of *simplicial map* $f : X \to Y$ between simplicial sets can be defined. The map f must be a system $\{f_m : X_m \to Y_m\}_{m \in \mathbb{N}}$ satisfying the commuting relations $\alpha_X^* \circ f_m = f_n \circ \alpha_Y^*$ if α is a Δ-morphism $\alpha \in \Delta(\underline{\mathbf{m}}, \underline{\mathbf{n}})$. If $f : X \to Y$ is such a simplicial map, a realization $|f| : |X| \to |Y|$, a continuous map, is canonically defined.

7.7 Associated chain-complexes.

In the same way simplicial *complexes* produce chain-complexes, see Section 2.2.3, simplicial sets also produce chain-complexes.

Definition 111 — Let X be a simplicial set. The *chain-complex* $C_*(X) = C_*(X; \mathfrak{R})$ *associated with* X is defined as follows:

- $C_m(X)$ is the free $\mathfrak{R}$-module generated by X_m, the set of m-simplices of X;
- The differential $d : C_m(X) \to C_{m-1}(X)$ is the $\mathfrak{R}$-linear morphism defined by $d(\sigma) = \sum_{i=0}^m (-1)^i \partial_i(\sigma)$ if $\sigma \in X_m$.

In algebraic topology, most often some ground ring $\mathfrak{R}$ is underlying, frequently $\mathfrak{R} = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{Z}_p$ for a prime p. The chain-complex so defined is the *standard* chain-complex, not taking account of possible shorthands due to *degenerate* simplices. From a theoretical point of view, this chain-complex is frequently more convenient, because the technicalities about the nature, degenerate or not, of every simplex are not necessary. On the contrary, for concrete calculations, typically for finite simplicial sets, the *normalized* associated chain-complex can be more convenient. The right statement of the Eilenberg-Zilber Theorem, see Section 8, also requires normalized chain-complexes, and it is so important this will become the *default* option.

Definition 112 — Let X be a simplicial set. The *normalized* chain-complex $C_*^N(X) = C_*^N(X; \mathfrak{R})$ associated with X is defined as follows:

- $C_m^N(X)$ is the free $\mathfrak{R}$-module generated by X_m^{ND}, the set of *non-degenerate* m-simplices of X;
- The differential $d : C_m^N(X) \to C_{m-1}^N(X)$ is the $\mathfrak{R}$-linear morphism defined by $d(\sigma) = \sum_{i=0}^m (-1)^i \partial_i(\sigma)$ if $\sigma \in X_m$ where every possible occurence of a degenerate simplex in the alternate sum is cancelled.

See page 8 where the minimal triangulation of the real projective plane, called `short-P2R` was used to compute the homology of this projective plane. The happy event is that for every simplicial set X, both chain-complexes $C_*(X)$ and $C_*^N(X)$ have the same homology.

Theorem 113 (Normalization Theorem) *The graded submodule $C_*^D(X)$ generated by the degenerate simplices is a sub*complex *of $C_*(X)$: the boundary of a degenerate simplex is a combination of degenerate simplices; this chain-complex is acyclic. The canonical isomorphism $C_*^N(X) \cong C_*(X)/C_*^D(X)$ induces a canonical isomorphism $H_*(C_*(X)) \cong H_*(C_*^N(X))$.*

The *right* definition of $C_*^N(X)$ in fact is $C_*^N(X) := C_*(X)/C_*^D(X)$. The inductive proof [36, VIII.6] can easily be arranged to prove:

Theorem 114 *A general algorithm computes:*

$$X \mapsto [\rho_X : C_*(X) \Rrightarrow C_*^N(X)]$$

where:

1. *X is a simplicial set;*
2. *ρ_X is a chain-complex reduction.*

If a simplex σ is an m-simplex, the induction can be chosen going from 0 to m or symmetrically from m to 0. So that there are *two* such canonical general algorithms. See also [57] for a *categorical* programming of this algorithm.

7.8 Products of simplicial sets.

Definition 115 — If X and Y are two simplicial sets, the *simplicial product* $Z = X \times Y$ is defined by $Z_m = X_m \times Y_m$ for every natural number m, and $\alpha_Z^* = \alpha_X^* \times \alpha_Y^*$ if α is a Δ-morphism.

The definition of the product of two simplicial sets is perfectly trivial and is however at the origin of several landmark problems in algebraic topology, for example the deep structure of the twisted Eilenberg-Zilber Theorem, still quite mysterious, and also the enormous field around the Steenrod algebras.

Every simplex of the product $Z = X \times Y$ is a *pair* (σ, τ) made of one simplex in X and one simplex in Y; both simplices must have the *same dimension.* It is tempting at this point, because of the "product" ambience, to denote by $\sigma \times \tau$ such a simplex in the product but *this would be a terrible error!* This is not at all the right point of view; the pair $(\sigma, \tau) \in Z_m$ is the unique simplex in Z whose respective *projections* in X and Y are σ and τ and this is the reason why the pair notation (σ, τ) is the only one which is possible. For example the diagonal of a square is a 1-simplex, the unique simplex the projections of which are both factors of the square; on the contrary, the "product" of the factors is simply the square, which does not have the dimension 1 and which is even not a simplex.

Theorem 116 — *If X and Y are two simplicial sets and $Z = X \times Y$ is their simplicial product, then there exists a canonical homeomorphism between $|Z|$ and $|X| \times |Y|$, the last product being the product of CW-complexes (or also of k-spaces).*

If you consider the product $|X| \times |Y|$ as the product of topological spaces, the same accident as for CW-complexes (see [35]) can happen.

Proof. There are natural simplicial projections $X \times Y \to X$ and Y which define a canonical continuous map $\phi : |X \times Y| \to |X| \times |Y|$. The interesting question is to define its inverse $\psi : |X| \times |Y| \to |X \times Y|$.

First of all, let us detail the case of $X = \Delta^2$ and $Y = \Delta^1$ where the essential points are visible. The first factor X has dimension 2, and the second one Y has dimension 1 so that the product Z shoud have dimension 3. What about the 3-simplices of Z? There are 3 such *non-degenerate* 3-simplices, namely $\rho_0 = (\eta_0\sigma, \eta_2\eta_1\tau)$, $\rho_1 = (\eta_1\sigma, \eta_2\eta_0\tau)$ and $\rho_2 = (\eta_2\sigma, \eta_1\eta_0\tau)$, if σ (resp. τ) is the unique non-degenerate 2-simplex (resp. 1-simplex) of Δ^2 (resp. Δ^1). This is nothing but the decomposition of a prism $\Delta^2 \times \Delta^1$ in three tetrahedrons.

Note no non-degenerate 3-simplex is present in X and Y and however some 3-simplices must be produced for Z; this is possible thanks to the *degenerate* simplices of X and Y where they are again playing a quite tricky role in our workspace; in particular a pair of *degenerate* simplices in the factors can produce a *non-degenerate* simplex in the product! This happens when there is no common degeneracy in the factors.

For example the tetrahedron $\rho_0 = (\eta_0\sigma, \eta_2\eta_1\tau)$ inside Z is *the* unique 3-simplex the first projection of which is $\eta_0\sigma$, and the second projection is $\eta_2\eta_1\tau$; the first projection is a tetrahedron collapsed on the triangle σ, identifying two points when the sum of barycentric coordinates of index 0 and 1 (the indices where injectivity fails in η_0) are equal; the second projection is a tetrahedron collapsed on an interval, identifying two points when the sum of barycentric coordinates of index 1, 2 and 3 are equal.

Let us take a point of coordinates $r = (r_0, r_1, r_2, r_3)$ in the simplex ρ_0. Its first projection is the point of $X = \Delta^2$ of barycentric coordinates $s = (s_0 = r_0 + r_1, s_1 = r_2, s_2 = r_3)$; in the same way its second projection is the point of $Y = \Delta^1$ of barycentric coordinates $t = (t_0 = r_0, t_1 = r_1 + r_2 + r_3)$. So that:

$$\phi(\rho_0, (r_0, r_1, r_2, r_3)) = ((\sigma, (r_0 + r_1, r_2, r_3)), (\tau, (r_0, r_1 + r_2 + r_3)))$$

In the same way:

$$
\begin{aligned}
\phi(\rho_1,(r_0,r_1,r_2,r_3)) &= ((\sigma,(r_0,r_1+r_2,r_3)),(\tau,(r_0+r_1,r_2+r_3))) \\
\phi(\rho_2,(r_0,r_1,r_2,r_3)) &= ((\sigma,(r_0,r_1,r_2+r_3)),(\tau,(r_0+r_1+r_2,r_3)))
\end{aligned}
$$

The challenge then consists in deciding for an arbitrary point $((\sigma,(s_0,s_1,s_2)),(\tau,(t_0,t_1))) \in |X| \times |Y|$ what simplex ρ_i it comes from and what a good ϕ-preimage (ρ_i,r) could be. You obtain the solution in comparing the sums $u_0 = s_0$, $u_1 = s_0 + s_1$, $u_2 = t_0$; the sums $s_0+s_1+s_2$ and t_0+t_1 are necesarily equal to 1 and do not play any role. You see in the three cases, the values of u_i's are:

$$
\begin{aligned}
((\eta_0\sigma,\eta_2\eta_1\tau),r) \Rightarrow&\quad u_0 = r_0+r_1, u_1 = r_0+r_1+r_2, u_2 = r_0, \\
((\eta_1\sigma,\eta_2\eta_0\tau),r) \Rightarrow&\quad u_0 = r_0, u_1 = r_0+r_1+r_2, u_2 = r_0+r_1, \\
((\eta_2\sigma,\eta_1\eta_0\tau),r) \Rightarrow&\quad u_0 = r_0, u_1 = r_0+r_1, u_2 = r_0+r_1+r_2,
\end{aligned}
$$

so that you can guess the degeneracy operators to be applied to the factors σ and τ from the order of the u_i's; more precisely, sorting the u_i's puts the u_2 value in position 0, 1 or 2, and this gives the index for the degeneracy to be applied to σ; in the same way the u_0 and u_1 values must be installed in positions "1 and 2", or "0 and 2", or "0 and 1" and this gives the two indices (in reverse order) for the degeneracies to be applied to τ. It's a question of *shuffle*. Furthermore you can find the components r_i from the differences between successive u_i's. Now we can construct the map ψ:

$$
\begin{aligned}
\phi((\sigma,s)(\tau,t)) &= (\rho_0,(u_2,u_0-u_2,u_1-u_0,1-u_1)) \quad \text{if } u_2 \le u_0 \le u_1, \\
&= (\rho_1,(u_0,u_2-u_0,u_1-u_2,1-u_1)) \quad \text{if } u_0 \le u_2 \le u_1, \\
&= (\rho_2,(u_0,u_1-u_0,u_2-u_1,1-u_2)) \quad \text{if } u_0 \le u_1 \le u_2.
\end{aligned}
$$

There seems an ambiguity occurs when there is an equality between u_2 and u_0 or u_1, but it is easy to see both possible preimages are in fact the same in $|Z|$.

Now this can be extended to the general case, according to the following recipe. Let $\sigma \in X_m$ and $\tau \in Y_n$ be two simplices, $s \in \Delta_m$ and $t \in \Delta^n$ two geometric points. We must define $\psi((\sigma,s),(\tau,t)) \in |Z| = |X \times Y|$. We set $u_0 = s_0$, $u_1 = s_0 + s_1$, ..., $u_{m-1} = s_0 + \ldots + s_{m-1}$, $u_m = t_0$, $u_{m+1} = t_0 + t_1$, ..., $u_{m+n-1} = t_0 + \ldots + t_{n-1}$. Then we sort the u_i's according to the increasing order to obtain a sorted list $(v_0 \le \ldots \le v_{m+n-1})$. In particular $u_m = v_{i_0}, \ldots, u_{m+n-1} = v_{i_{n-1}}$ with $i_0 < \ldots < i_{n-1}$, and $u_0 = v_{j_0}, \ldots, u_{m-1} = v_{j_{m-1}}$ with $j_0 < \ldots < j_{m-1}$. Then:

$$
\begin{aligned}
&\psi((\sigma,s),(\tau,t)) = \\
&((\eta_{i_{n-1}}\ldots\eta_{i_0}\sigma,\eta_{j_{m-1}}\ldots\eta_{j_0}\tau),(v_0,v_1-v_0,\ldots,v_{m+n-1}-v_{m+n-2},1-v_{m+n-1})).
\end{aligned}
$$

Now it is easy to prove $\psi \circ \phi = \mathrm{id}_{|Z|}$ and $\phi \circ \psi = \mathrm{id}_{|X|\times|Y|}$, following the proof structure clearly visible in the case of $X = \Delta^2$ and $Y = \Delta^1$.

It is also necessary to prove the maps ϕ and ψ are continuous. But ϕ is the product of the realization of two simplicial maps and is therefore continuous. The map ψ is defined in a coherent way for each *cell* $\sigma \times \tau$ (this time it is really the *product* $|\sigma| \times |\tau| \subset |X| \times |Y|$) and is clearly continuous on each cell; because of the definition of the CW-topology, the map ψ is continuous. ■

If three simplicial sets X, Y and Z are given, there is only one natural map $|X \times Y \times Z| \to |X| \times |Y| \times |Z|$ so that "both" inverses you construct by applying twice the previous construction of ψ, the first one going through $|X \times Y| \times |Z|$, the second one through $|X| \times |Y \times Z|$ are necessarily the same: the ψ-construction is *associative*, which is interesting to prove directly; it is essentially the associativity of the Eilenberg-MacLane formula.

7.8.1 The case of simplicial groups.

Let G be a *simplicial group*. The object G is a simplicial object in the group category; in other words each simplex set G_m is provided with a group structure and the Δ-operators $\alpha^* : G_m \to G_n$ are group homomorphisms.

This gives in particular a continuous canonical map $|G \times G| \to |G|$; then identifying $|G \times G|$ and $|G| \times |G|$, we obtain a "continuous" group structure for $|G|$; the word *continuous* is put between quotes, because this does not work in general in the topological sense: this works always only in the category of "CW-groups" where the group structure is a map $|G| \times |G| \to |G|$, the source of which being evaluated in the CW-category; because of this definition of product, it is then true that $|G| \times |G| = |G \times G|$. The composition rule so defined on $|G|$ satisfies the group axioms; in particular the associativity property comes from the considerations about the associativity of the ψ-construction in the previous section.

7.9 Kan extension condition.

Let us consider the standard simplicial model S^1 of the circle, with one vertex and one non-degenerate 1-simplex σ. This unique 1-simplex clearly represents a generator of $\pi_1(S^1)$, but its double cannot be so represented. This has many disadvantages and correcting this defect was elegantly solved by Kan.

Definition 117 — A *Kan* (m, i)*-hat* (Kan hat in short) in a simplicial set X is a $(m+1)$-tuple $(\sigma_0, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{m+1})$ satisfying the relations $\partial_j \sigma_k = \partial_{k-1}\sigma_j$ if $j < k$, $j \neq i \neq k$.

For example the pair $(\partial_0\mathbf{id}, \partial_1\mathbf{id}, \partial_2\mathbf{id},)$ is a Kan $(3, 3)$-hat in the standard 3-simplex Δ^3 if $\mathbf{id}$ is the unique non-degenerate 3-simplex. Also the pair (σ, σ) is a Kan $(2, 1)$-hat of the above S^1.

Definition 118 — If $(\sigma_0, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{m+1})$ is a Kan (m, i)-hat in the simplicial set X, a *filling* of this hat is a simplex $\sigma \in X_{m+1}$ such that $\partial_j \tau = \sigma_j$ for $j \neq i$.

The 3-simplex **id** of Δ^3 is a filling of the example Kan hat in Δ^3. The example Kan hat of S^1 has no filling. A Kan (m, i)-hat is a system of m-simplices arranged like all the faces except the i-th one of a *hypothetical* $(m + 1)$-simplex.

Definition 119 — A simplicial set X satisfies the *Kan extension condition* if any Kan hat has a filling.

The standar simplex Δ^d satisfies the Kan condition. The other elementary simplicial sets in general do not.

The simplicial sets satisfying the Kan extension condition have numerous interesting properties; for example their homotopy groups can be combinatorially defined [40, Chapter 1], a canonical *minimal* version is included, also satisfying the extension condition [40, Section 9], a simple decomposition process produces a Postnikov tower [40, Section 8].

The simplicial groups are important in this respect: in fact a simplicial group always satisfies the Kan extension condition [40, Theorem 17.1]. For example the simplicial description of $P^\infty\mathbb{R}$ (see Section 7.6.1) is a simplicial group and therefore satisfies the Kan condition, which is not so obvious; it is even minimal. The singular complex SX of a topological space X also satisfies the Kan condition but in general is not minimal. These simplicial sets satisfying the Kan condition are so interesting that it is often useful to know how to *complete* an arbitrary given simplicial set X and produce a new simplicial set X' with the same homotopy type satisfying the Kan condition. The Kan-completed X' can be constructed as follows.

Let us define first an elementary completion $\chi(X)$ for X. For each Kan (m, i)-hat of X, we decide to add the hypothetical $(m + 1)$-simplex (even if a "solution" preexists), and the "missing" i-th face; such a completion operation does not change the homotopy type of X. Doing this completion construction for every Kan hat of X, we obtain the first completion $\chi(X)$. Then we can define $X_0 = X$, $X_{i+1} = \chi(X_i)$ and $X' = \lim_\rightarrow X_i$ is the desired Kan completion. You can also run this process in considering only the failing hats.

7.10 Simplicial fibrations.

A *fibration* is a map $p : E \to B$ between a *total space* E and a *base space* B satisfying a few properties describing more or less the total space E as a *twisted product* $F \times_\tau B$. In the simplicial context, several definitions are possible. The notion of *Kan* fibration corresponds to a situation where a simplicial homotopy lifting property is satisfied; to state this property, the elementary datum is a Kan hat in the total space and a given filling of its projection in the base space; the Kan fibration property is satisfied if it is possible to fill the Kan hat in the total space in a coherent way with respect to the given filling in the base space. This notion is the simplicial version of the notion of *Serre fibration*, a projection where the homotopy lifting property is satisfied for the maps defined on polyhedra. The reference [40] contains a detailed study of the basic facts around Kan fibrations, see [40, Chapters I and II].

We will examine with a little more details the notion of *twisted cartesian product*, corresponding to the topological notion of fibre bundle. It is a key notion in topology, and the simplicial framework is particularly favourable for several reasons. In particular the Serre spectral sequence becomes well structured in this environment, allowing us to extend it up to a *constructive* version, one of the main subjects of another lecture series of this Summer School. We give here the basic necessary definitions for the notion of twisted cartesian product.

A reasonably general situation consists in considering the case where a simplicial group G acts on the fibre space, a simplicial set F, the fibre space. As usual this means a map $\phi : F \times G \to F$ is given; source and target are simplicial sets, the first one being the product of F by the simplicial set G, underlying the simplicial group; the map ϕ is a simplicial map; furthermore each component $\phi_m : (F \times G)_m = F_m \times G_m \to F_m$ must satisfy the traditional properties of the right actions of a group on a set. We will use the shorter notation $f.\,g$ instead of $\phi(f, g)$. Let also B be our base space, some simplicial set.

Definition 120 — A *twisting operator* $\tau : B \to G$ is a family of maps $\{\tau_m : B_m \to G_{m-1}\}_{m\geq 1}$ satisfying the following properties.

1. $\partial_0\tau(b) = \tau(\partial_1 b)\tau(\partial_0 b)^{-1}$;
2. $\partial_i\tau(b) = \tau(\partial_{i+1}(b))$ if $i \leq 1$;
3. $\eta_i\tau(b) = \tau(\eta_{i+1}b)$;
4. $\tau(\eta_0 b) = e_m$ if $b \in G_{m+1}$, the unit element of G_m being e_m.

In particular it is not required τ is a *simplicial map*, and in fact, because of the degree -1 between source and target dimensions, this does not make sense.

Definition 121 — If a twisting operator $\tau : B \to G$ is given, the corresponding *twisted cartesian product* $E = F \times_\tau B$ is the simplicial set defined as follows. Its set of m-simplices E_m is the same as for the non-twisted product $E_m = F_m \times B_m$; the face and degeneracy operators are also the same as for the non-twisted product with only one exception: $\partial_0(f, b) = (\partial_0 f.\,\tau(b), \partial_0 b)$.

The twisting operator τ, the unique ingredient at the origin of a difference between the non-twisted product and the τ-twisted one, acts in the following way: the twisted product is constructed in a recursive way with respect to the base dimension. Let $B^{(k)}$ be the k-skeleton of B and let us suppose $F \times_\tau B^{(k)}$ is already constructed. Let σ be a $(k+1)$-simplex of B; we must describe how the product $F \times \sigma$ is to be attached to $F \times B^{(k)}$; what is above the faces $\partial_i\sigma$ for $i \geq 1$ is naturally attached; but what is above the 0-face is shifted by the translation defined by the operation of $\tau(b)$. It is not obvious such an attachment is coherent, but the various formulas of Definition 121 are exactly the relations which must be satisfied by τ for consistency. It was not obvious, starting from scratch, to guess this is a good framework for working simplicially about fibrations; this was invented (discovered ?) by Daniel Kan [32]; the previous work by Eilenberg and MacLane [20, 21] in the particular case of the fibrations relating the elements of the Eilenberg-MacLane spectra was probably determining.

7.10.1 The simplest example.

Let us describe in this way the exponential fibration $\mathbf{exp} : \mathbb{R} \to S^1 : t \to e^{2\pi i t}$. We take for S^1 the model with one vertex $*_0$ and one non-degenerate edge $\mathrm{id}(\underline{\mathbf{1}}) = \sigma$ (see Section 7.5.3). For $\mathbb{R}$, we choose $\mathbb{R}_0 = \mathbb{Z}$ and $\mathbb{R}_1^{ND} = \mathbb{Z}$, that is one vertex k_0 and one non-degenerate edge k_1 for each integer $k \in \mathbb{Z}$; the faces are defined by $\partial_i(k_1) = (k+i)_0$ ($i = 0$ or 1). The discrete (see Section 7.5.1) simplicial group $\mathbb{Z}$ acts on the fibre; for any dimension d, the simplex group $\mathbb{Z}_d$ is $\mathbb{Z}$ with the natural structure, and $k_i \,.\, g = (k+g)_i$ for $i = 0$ or 1. It is then clear that the right twisting operator for the exponential fibration is $\tau(g) = 1$ for $g \in \mathbb{R}_1^{ND}$.

7.10.2 Fibrations between $K(\pi, n)$'s.

Let us recall (see Section 7.5.5) $E(\pi, d)$ is the simplicial set defined by $E(\pi, d)_m = C^d(\Delta^m, \pi)$ (only *normalized* cochains) and $K(\pi, n)$ is the simplicial subset made of the *cocycles*. The maps between simplex sets to be associated with Δ-morphisms are naturally defined. A simplicial projection $p : E(\pi, d) \to K(\pi, d+1)$ associating to an m-cochain c its coboundary δc, necessarily a cocycle, is also defined. The simplicial set Δ^m is contractible, its cochain-complex is acyclic and the kernel of p, the potential *fibre space*, is therefore the simplicial set $K(\pi, d)$. The base space is clearly the quotient of the total space by the fibre space (*principal* fibration), and a systematic examination of such a situation (see [40, Section 18]) shows $E(\pi, d)$ is necessarily a twisted cartesian product of the base space $K(\pi, d+1)$ by the fibre space $K(\pi, d)$.

It is not so easy to guess a corresponding twisting operator. A solution is described as follows; let $z \in Z^{d+1}(\Delta^m, \pi)$ a base m-simplex; the result $\tau(z) \in Z^d(\Delta^{m-1}, \pi)$ must be a d-cocyle of Δ^{m-1}, that is a function defined on every $(d+1)$-tuple $(i_0, \ldots, i_d)$, with values in π, and satisfying the cocycle condition; the solution $\tau(z)(i_0, \ldots, i_d) = z(0, i_0+1, \ldots, i_d+1) - z(1, i_0+1, \ldots, i_d+1)$ works, but seems a little mysterious. The good point of view consists in considering the notion of *pseudo-section* for the studied fibration; an actual section cannot exist if the fibration is not trivial, but a pseudo-section is essentially as good as possible; see the definition of pseudo-section in [40, Section 18]. When a pseudo-section is found, a simple process produces a twisting operator; in our example, the twisting operator comes from the pseudo-section $\rho(z)(i_0, \ldots, i_d) = z(0, i_0+1, \ldots, i_d+1)$, quite natural.

The fibrations between Eilenberg-MacLane spaces are a particular case of universal fibrations associated with simplicial groups. See [40, Section 21].

7.10.3 Simplicial loop spaces.

A simplicial set X is *reduced* if its 0-simplex set X_0 has only one element. We have given in Section 7.5.6 the Kan combinatorial version GX of the loop space of X. This loop space is the fibre space of a *co-universal* fibration:

$$GX \hookrightarrow GX \times_\tau X \to X.$$

Only the twisting operator τ remains to be defined. The definition is simply... $\tau(\sigma) := \tau(\sigma)$ for both possible meanings of $\tau(\sigma)$; the first one is the value of the twisting operator to be defined for some simplex $\sigma \in X_{m+1}$ and the second one is the generator of GX_m corresponding to $\sigma \in X_{m+1}$, the unit element of GX_m if ever σ is 0-degenerate (see Section 7.5.6). The definition of the face operators for GX are exactly those which are required so that the twisting operator so defined is coherent.

It is again an example of *principal fibration*, that is the fibre space is equal to the structural group and the action $GX \times GX \to GX$ is equal to the group multiplication. This fibration is co-universal, with respect to X; in fact, let $H \hookrightarrow H \times_{\tau'} X \xrightarrow{p} X$ another *principal* fibration above X for another twisting operator $\tau' : X \to H$. Then the free group structure of GX gives you a unique group homomorphism $GX \to H$ inducing a canonical morphism between both fibrations.

If the simplicial space X is 1-reduced (only one vertex, no non-degenerate 1-simplex), then an important result by John Adams [1] allows one to compute the homology groups of GX if the initial simplicial set X is of finite type; an intermediate ingredient, the *Cobar construction*, is the key point. One of the main problems in Algebraic Topology consists in solving the analogous problem for the iterated loop spaces $G^n X$ when X is n-reduced; it is the problem of *iterating the Cobar construction*; one of the lecture series of this Summer School is devoted to this subject.

8 Serre spectral sequence.

8.1 Introduction.

We begin now the part of this text devoted to Algebraic Topology. The general idea is that Topology is difficult; on the contrary, Algebra is easy, an appreciation certainly shared by Deligne, Faltings, Wiles, Lafforgue... Let us be serious; as explained in the introduction of this text, the matter is not at all to switch from Topology to Algebra, the actual subject is to make topology *constructive*, in particular the natural problem of *classification*. Because common algebra has a naturally constructive framework, it is understood switching from topology to algebra could be useful. The goal of *constructive* algebraic topology consists in organizing the translation process in such a way that common constructive algebra actually allows you to constructively work in topology.

The Eilenberg-Zilber Theorem is unavoidable in Algebraic Topology, it allows to compute $H_*(X \times Y)$ when $H_*(X)$ and $H_*(Y)$ are known. In a sense it is the last case where "ordinary" algebraic topology succeeds: *ordinary* homology groups of the ingredients X and Y are sufficient to determine the homology groups of the product $X \times Y$. The next natural case concerns the Serre spectral sequence: if the product $X \times_\tau Y$ is *twisted*, then the ordinary homology groups of X and Y in general are not sufficient to start an algorithm determining $H_*(X \times_\tau Y)$.

We present in this section both Eilenberg-Zilber Theorems, the original one, non-twisted, and also the twisted one, in fact due to Edgar Brown [10], put under its modern form by Shih Weishu [61] and Ronnis Brown [11]. The *effective* Serre spectral sequence is then an obvious consequence of the *twisted* Eilenberg-Zilber Theorem.

UOStated 122 — *In the part of this text devoted to Algebraic Topology, if X is a simplicial set, $C_*(X)$ denotes the* normalized *chain-complex $C_*^N(X)$ canonically associated with X; that is, $C_*(X)$ denotes which should be denoted by $C_*^N(X) := C_*(X)/C_*^D(X)$.*

Because of Theorem 114, this choice has no incidence upon the theoretical nature of the results. For concrete calculations, one or other choice can significantly change computing time and/or space.

8.2 The Eilenberg-Zilber Theorem.

If X and Y are two simplicial sets, the *cartesian product* $X \times Y$ is naturally defined by $(X \times Y)_n = X_n \times Y_n$, and the face and degeneracy operators are the products of the corresponding operators of each factor simplicial set; see Definition 115. If $\sigma \in X_n$ and $\tau \in Y_n$ are two n-simplices, the notation (σ, τ) must be preferred to the tempting notation $\sigma \times \tau$: the pair notation (σ, τ) has the advantage to clearly mean this is the n-simplex whose first (resp. second) *projection* is σ (resp. τ). The "product" $\sigma \times \tau$, even if both simplices have not the same dimension, should normally denote the element of $C_*(X \times Y)$ which is the Eilenberg-MacLane image of the element $\sigma \otimes \tau \in C_*X \otimes C_*Y$, that is, the geometrical decomposition in simplices of the geometrical product of σ and τ.

Theorem 123 (Eilenberg-Zilber Theorem)— *A general algorithm computes:*

$$(X, Y) \mapsto [\rho_{X,Y} : C_*(X \times Y) \Rrightarrow C_*(X) \otimes C_*(Y)].$$

where:

1. *X and Y are simplicial sets;*
2. *$\rho_{X,Y}$ is a* reduction *from the chain-complex of the product $C_*(X \times Y)$ over the tensor product of chain-complexes $C_*(X) \otimes C_*(Y)$.*

Let us recall this theorem requires considering *normalized* chain-complexes. It is frequently presented as a consequence of the theorem of acyclic models [62], which is not very explicit; however this method can be made effective [51]. It is simpler to use the effective formulas for the Eilenberg-Zilber reduction $\rho_{X,Y} = (f, g, h)$ known as the Alexander-Whitney (f), Eilenberg-MacLane (g) and Shih (h) operators. They come from the the recursive definition of these operators (see [20] and [21], or [61]). It is in the papers [20, 21] that (homological) reductions[25] between chain-complexes appeared for the first time. Only the last requirement $h^2 = 0$ was missing.

The Eilenberg-MacLane and Shih operators have an essential "exponential" nature. It is not a question of method of computation, it is a question of very nature: the number of *different terms* produced by the Eilenberg-MacLane operator working on a tensor product of bi-degree (p, q) is the binomial coefficient $\binom{p+q}{p}$. So that any algorithm going through such a formula is necessarily of exponential complexity. Furthermore this formula is unique [48], and the difficulty localized here is therefore quite essential. In a sense, "classical" algebraic topology, typically the work around Steenrod operations, consists in avoiding the definitively exponential complexity of the Eilenberg-MacLane formula in order to be able to reach high dimensions; this text on the contrary focuses on *arbitrary* spaces in low dimensions (something like < 12) where much interesting work is also to be done. A consequence of these considerations is that our computing methods will certainly not lead to high sphere homotopy groups; we are processing the *orthogonal* problem: we are not concerned by high dimensional invariants of *known* objects, we are only interested by the first invariants of *random* objects.

Interpreting the Eilenberg-Zilber Theorem in the framework of objects with effective homology requires *composition* of equivalences.

Proposition 124 — *A general algorithm computes:*

$$[\varepsilon : A_* \lll B_* \ggg C_*, \varepsilon' : C_* \lll D_* \ggg E_*] \mapsto \varepsilon'' : [A_* \lll F_* \ggg E_*]$$

where:

1. *ε and ε' are two given equivalences between chain-complexes, the "target" of ε being the "source" of ε'.*
2. *ε'' is an equivalence between the extreme chain-complexes which must be considered as the composition $\varepsilon'' = \varepsilon \circ \varepsilon'$.*

Proof. Instead of a complex direct proof, a small collection of quite elementary lemmas gives the answer.

Lemma 125 — *The cone of an identity chain map* $\mathrm{id} : C_* \leftarrow C_*$ *is acyclic; more precisely a simple algorithm constructs a reduction* $\mathrm{Cone}(\mathrm{id}) \ggg 0$.

Proof. Apply Lemma 82 to the short exact sequence:

$$0 \leftarrow 0 \leftarrow C_* \xleftarrow{\mathrm{id}} C_* \leftarrow 0.$$

■

[25] They were called *contractions*, but it was a serious terminological imprecision: reduction is reserved for simplification in an *algebraic* framework, and contraction in a *topological* framework. And it is essential to understand that a chain-complex associated with a topological object in general loses the topological nature of the object.

Lemma 126 — *Let $\rho = (f, g, h) : D_* \Rrightarrow C_*$ be a reduction. Then Cone(f) is acyclic; more precisely, an algorithm constructs a reduction Cone(f) $\Rrightarrow 0$.*

Proof. Applying the Cone Reduction Theorem 62 to $\mathrm{Cone}(f)$, using the given reduction ρ for the source D_* over C_* and the trivial identity reduction $C_* \Rrightarrow C_*$ for the target C_* produces a reduction $\mathrm{Cone}(f) \Rrightarrow \mathrm{Cone}(\mathrm{id}_{C_*})$. Composing (Proposition 59) this reduction with the reduction $\mathrm{Cone}(\mathrm{id}_{C_*}) \Rrightarrow 0$ of the previous lemma gives the result. ∎

Definition 127 — If $f : B_* \to C_*$ and $f' : C_* \leftarrow D_*$ are two chain-complex morphisms, the *bicone* $\mathrm{BiCone}(f, f')$ is constructed from $\mathrm{Cone}(f)$ and $\mathrm{Cone}(f')$ by identification of both target chain-complexes C_*.

It is an *algamated* sum of both cones *along* the common component C_*.

Lemma 128 — *Let $\rho = (f, g, h) : B_* \Rrightarrow C_*$ and $\rho' = (f', g', h') : D_* \Lleftarrow C_*$ be two reductions. An algorithm constructs a reduction* $\mathrm{BiCone}(f, f')^{[-1]} \Rrightarrow B_*$ *and another one* $\mathrm{BiCone}(f, f')^{[-1]} \Rrightarrow D_*$.

Proof. The bicone $\mathrm{BiCone}(f, f')$ can be interpreted as $\mathrm{Cone}(f : B_* \to \mathrm{Cone}(f'))$, calling again f the chain-complex morphism with the same source as f and going to C_* which is also a sub-*chain-complex* of $\mathrm{Cone}(f')$. This allows us to apply again the Cone Reduction Theorem to the trivial identity reduction over B_* and the reduction to 0 of $\mathrm{Cone}(f')$. The desuspension process for the bicone is necessary, because in a cone, the source is suspended. ∎

Proof of Proposition 124. It is a consequence of the next diagram and composition of reductions:

$$A_* \Lleftarrow B_* \Lleftarrow \mathrm{BiCone}(f, f')^{[-1]} \Rrightarrow D_* \Rrightarrow E_*.$$

∎

Corollary 129 — *A general algorithm computes:*

$$(X_{EH}, Y_{EH}) \mapsto (X \times Y)_{EH}$$

where:

1. *X_{EH} and Y_{EH} are simplicial sets* with effective homology*;*
2. *$(X \times Y)_{EH}$ is a version* with effective homology *of the product $X \times Y$.*

Proof. Let $(X, C_*(X), EC_X, \varepsilon_X)$ and $(Y, C_*(Y), EC_Y, \varepsilon_Y)$ be two simplicial sets with effective homology. Eilenberg and Zilber give an equivalence ε_1 : $C_*(X \times Y) \overset{=}{\Lleftarrow} C_*(X \times Y) \Rrightarrow C_*(X) \otimes C_*(Y)$ (the left reduction is trivial); Proposition 60 gives also an equivalence ε_2 between $C_*(X) \otimes C_*(Y)$ and $EC_X \otimes EC_Y$. Composing these homotopy equivalences (Proposition 124), we obtain the wished homotopy equivalence between $C_*(X \times Y)$ and the *effective* chain-complex $EC_X \otimes EC_Y$ ■

The Künneth Theorem is not used; it allows you to *guess* the homology groups of $EC_X \otimes EC_Y$ if you know the homology groups of factors, but we are not concerned by this question: the chain-complexes EC_X and EC_Y are effective, so that $EC_X \otimes EC_Y$ is also effective, and this is sufficient. We are on the contrary essentially interested by an *explicit* homotopy equivalence between $C_*(X \times Y)$ and $EC_X \otimes EC_Y$, and the explicit definition of the Eilenberg-Zilber reduction is the key point.

Let us finish this presentation of the Eilenberg-Zilber Theorem by a typical application. It is elementary to compute the homology of the real projective plane $P^2\mathbb{R}$; this was done by Kenzo page 9, but once the simplicial set technique is known, pen and paper are enough. The minimal[26] simplicial descrition has three non-degenerate simplices: one vertex, the base point, one edge, the equivalence class of the equator and one triangle. The normalized chain-complex is:

$$C_*(P^2\mathbb{R}) = [\cdots 0 \leftarrow \mathbb{Z} \stackrel{0}{\leftarrow} \mathbb{Z} \stackrel{\times 2}{\leftarrow} \mathbb{Z} \leftarrow 0 \cdots]$$

with $\times 2$ between degrees 2 and 1.

If you work in the style of traditional algebraic topology, you deduce the homology groups $H_*(P^2\mathbb{R}) = (\mathbb{Z}, \mathbb{Z}_2)$ in degrees 0 and 1, the others being null.

Now your client orders a *construction* $X := P^2\mathbb{R} \times P^2\mathbb{R}$ and asks for $H_*(X) =??$. In traditional style, you will try to deduce the homology groups of X from those of $P^2\mathbb{R}$; the answer is the Künneth formula [62, Section 5.3]:

$$H_n(X \times Y) = \left(\oplus_{p=0}^{n} H_p(X) \otimes H_{n-p}(Y)\right) \oplus \left(\oplus_{p=0}^{n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(X), H_{n-1-p}(Y))\right).$$

The bad student forgets the torsion terms, which require some lucidity. Furthermore the sum decomposition is not canonical. The result is $H_*(X) = (\mathbb{Z}, \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2)$, the last $\mathbb{Z}_2$ being the only contribution of torsion terms.

In the spirit of effective homology, you observe the normalized chain-complex $C_*(P^2\mathbb{R})$ is already effective, so that a trivial effective homology is enough. If ever you are interested by the standard homology groups, you can ask a machine, but here it is obvious, you obtain the right homology groups of $P^2\mathbb{R}$. Now what about $H_*(X)$? First you *simplicially* construct $X = P^2\mathbb{R} \times P^2\mathbb{R}$; it is not so simple, but a machine does it automatically; the numbers of non-degenerate simplices are $(1, 3, 9, 12, 6)$; the corresponding normalized chain-complex $C_*(X)$ is relatively complex, but Eilenberg and Zilber explain to us there is a reduction $C_*(X) \Rrightarrow C_*(P^2\mathbb{R}) \otimes C_*(P^2\mathbb{R})$ and the last chain-complex is elementarily com-

[26] Minimal by the number of simplices, but the Kan condition, see Section 7.9, is not satisfied, so that this minimal description of $P^2\mathbb{R}$ is not minimal in the sense of Kan [40, §9].

puted; presented as a bicomplex, it is:

$$\begin{tikzcd}
\mathbb{Z} & \mathbb{Z} \arrow[l, "0"'] \arrow[d, "\times(-2)"] & \mathbb{Z} \arrow[l, "\times 2"'] \arrow[d, "\times 2"'] \\
\mathbb{Z} \arrow[u, leftarrow, "\times 2"'] \arrow[d, "0"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, "0"'] & \mathbb{Z} \arrow[l, "\times 2"'] \arrow[d, "0"'] \\
\mathbb{Z} & \mathbb{Z} \arrow[l, "0"'] & \mathbb{Z} \arrow[l, "\times 2"']
\end{tikzcd}$$

giving the expected homology groups. This is nothing but the standard calculation giving $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, so that you can wonder why such a presentation? The crucial point is the following: your client will probably tomorrow undertake a new construction, more or less complicated, using the *geometry* of $X = (P^2\mathbb{R})^2$; if the construction is a little complicated, then it is not possible to *describe* it at the homological level and you cannot continue. We will soon see critical examples. In effective homology, the simplicial model of X remains present in your environment with the Eilenberg-Zilber connection with the chain-complex displayed above. Whatever construction is imagined by your client, *you will be ready to provide an algorithm* providing the effective homology of the resulting object.

8.3 The twisted Eilenberg-Zilber Theorem.

Let $F \hookrightarrow [E = F \times_\tau B] \to B$ be a twisted product, that is, a simplicial fibration defined by a *base space* B, some simplicial set, a *fibre space* F, another simplicial set, and some twisting operator $\tau : B \to G$, the target G being some simplicial group acting over the fibre space F. See Section 7.10 for details and examples. If τ is trivial, the Eilenberg-Zilber Theorem gives a reduction $C_*(E) \Rrightarrow C_*(F) \otimes C_*(B)$. The so-called "twisted" Eilenberg-Zilber Theorem constructs an analogous reduction $C_*(E) \Rrightarrow C_*(F) \otimes_t C_*(E)$, the index of $\otimes_t$ meaning the *differential* of the usual chain-complex tensor product $C_*(F) \otimes C_*(B)$ being (deeply) modified.

Theorem 130 (Twisted Eilenberg-Zilber Theorem) — *An algorithm computes:*

$$\Phi \mapsto \rho$$

where:

1. *Φ is a simplicial fibration $\Phi = \{F \hookrightarrow [E = F \times_\tau B] \to B\}$.*
2. *$\rho : C_*(E) \Rrightarrow C_*(F) \otimes_t C_*B$ is a reduction of the (normalized) chain-complex of the total space of the fibration over a chain-complex $C_*(F) \otimes_t C_*B$; the underlying* graded module *of the latter is the same as for $C_*(F) \otimes C_*B$, but the differential is modified to take account of the twisting operator τ.*

Proof. The ordinary (non-twisted) Eilenber-Zilber Theorem gives a reduction between the non-twisted cartesian and tensor products, the twisting operator being null. But we must take account of the twisting operator τ; this twisting operator does not change the underlying top graded module, only the differential is modified: the 0-face operator is twisted, see Definition 121. The basic perturbation lemma may be applied if the nilpotency condition is satisfied.

If (f, b) is a simplex of E, the component b has a unique form $b = \eta b'$ where b' is non-degenerate and η is a multy-degeneracy operator; if b is non-degenerate then $b' = b$ and η is the identity, no degeneracy at all. Following Serre, the filtration degree of (f, b) is the dimension of b', the "base dimension". The Shih homotopy operator of Eilenberg-Zilber is *natural*, and when it works on (f, b) it is equal to the one which is defined on $F \times b'$, just above the simplex b'; therefore the Shih operator does not increase the filtration degree.

On the contrary the perturbation $\widehat{\delta}(f, b) = (\partial_0 f . \tau(b), \partial_0 b) - (\partial_0 f, \partial_0 b)$ has a filtration degree smaller than the filtration degree of (f, b). If b is non-degenerate, it is obvious. If b is degenerate and if the η in the expression $b = \eta b'$ does not contain a η_0, then $\partial_0 \eta b' = \eta' \partial_0 b'$, because of the commuting relation $\partial_0 \eta_i = \eta_{i-1} \partial_0$ if $i > 0$; the filtration degree of $(f', \partial_0 b)$ is again less than the one of (f, b). Finally, if the multi-degeneracy operator η contains a η_0, then $\tau(b)$ is trivial, see Definition 120, and the perturbation is null. The nilpotency hypothesis is satisfied.

The basic perturbation lemma is then applied and produces the wished reduction. ■

The following technical proposition is the key point allowing one to use the twisted Eilenberg-Zilber Theorem to obtain a version with effective homology of the Serre spectral sequence.

Proposition 131 — *Let $\Phi = (B, F, G, \tau, E) = \{F \hookrightarrow [E = (F \times_\tau B)] \to B\}$ be a simplicial fibration. Let $\rho : C_*(F \times B) \Rightarrow C_*(F) \otimes C_*(B)$ (resp. $\rho' : C_*(F \times_\tau B) \Rightarrow C_*(F) \otimes_t C_*(B)$) be the non-twisted (resp. twisted) reduction given by the Eilenberg-Zilber (resp. twisted Eilenberg-Zilber) Theorem. Let d (resp. d') be the differential of $C_*(F) \otimes C_*(B)$ (resp. $C_*(F) \otimes_t C_*(B)$) and let $\delta = d' - d$ be the bottom differential perturbation computed by the twisted Eilenberg-Zilber Theorem. Then,* if B is 1-reduced*, the bottom perturbation δ decreases the filtration degree at least by 2.*

A simplicial set B is *1-reduced* if it has only one vertex and no non-degenerate 1-simplex, therefore, only one 1-simplex, the unique degeneracy of the unique vertex.

The conclusion of the proposition is to be understood as follows: if b (resp. f) is a p-simplex (resp. q-simplex) of B (resp. F), then:

$$\delta(f \otimes b) = \sum_{r=2}^{p} \delta_r(f \otimes b)$$

where $\delta_r(f \otimes b) \in C_{q+r-1}(F) \otimes C_{p-r}(B)$. Note it is not possible to coherently choose one of both possible notations $(f \otimes b)$ and $(f \otimes_t b)$: in fact $\delta = d' - d$ and d (resp. d') is to be applied to $(f \otimes b)$ (resp. $(f \otimes_t b)$).

Proof. Let $\rho = (AW, EML, SH)$ the ordinary Eilenberg-Zilber reduction between $C_*(F \times B)$ and $C_*(F) \otimes C_*(B)$. If $\widehat{\delta} = \widehat{d}' - \widehat{d}$ is the top perturbation, the explicit formula for the bottom perturbation in the proof of Theorem, cf. page 49, gives:

$$\delta(f \otimes b) = (AW \circ (\sum_{i=0}^{\infty} (-1)^i (\widehat{\delta} \circ SH)^i) \circ \widehat{\delta} \circ EML)(f \otimes b).$$

We have observed in the previous proof the top perturbation $\widehat{\delta}$ decreases the filtration degree at least by 1; furthermore, the Shih operator does not increase this filtration degree; therefore, the components with $i \geq 1$ in the expression just above satisfy the wished condition. The main work concerns only the $i = 0$ component.

The Eilenberg-MacLane operator working on $f \otimes b$ (f a non-degenerate q-simplex of F, b a non-degenerate p-simplex of B) produces a set of terms, shuffles of the form $\pm(\eta f, \eta' b)$ for some multi-degeneracy operators η and η'. If η' contains a η_0, then the corresponding twist is trivial and there is no perturbation. We can organize the other terms as follows: $\pm(\eta f, \eta' \eta'' b)$ where η contains a η_0, η'' is a composition of consecutive degeneracies $\eta'' = \eta_k \eta_{k-1} \ldots \eta_2 \eta_1 = \eta_1^k$, and η' is another composition $\eta' = \eta_{i_\ell} \ldots \eta_{i_1}$ with $i_1 \geq k+2$ and $k + \ell = q$; the integer $k+1$ is the first missing index in the degeneracies of the second component. We have then the expression:

$$(\widehat{\delta} \circ EML)(f \otimes b) = \sum \pm[(\partial_0 \eta f.\, \tau(\eta' \eta'' b), \eta'_{-1} \eta''_{-1} \partial_0 b) - (\partial_0 \eta f, \eta'_{-1} \eta''_{-1} \partial_0 b)].$$

In the expression above, a term η'_{-1} denotes the multi-degeneracy operator η' where all the indices have been replaced by the same minus one; in particular $\eta''_{-1} = \eta_{k-1} \ldots \eta_0$. There remains to apply the Alexander-Whitney operator:

$$AW(f', b') = \sum_{j=0}^{p+q-1} \partial_{j+1}^{p+q-1-j} f' \otimes \partial_0^j b'.$$

If $j > k$, then there are at least two operators ∂_0 which remain alive in the right component; this comes from the relation $\partial_0^j \eta_{k-1} \ldots \eta_0 = \partial_0^{j-k}$. In such a case, the term becomes something like $\pm(\ldots, \eta''' \partial_0^m b)$ with $m \geq 2$, and the result is obtained.

If $j \leq k$, the twisting modifier $\tau(\eta' \eta'' b)$ becomes by Alexander-Whitney $\tau(\partial_{j+2}^{p+q-1-j} \eta' \eta'' b)$, because the face index is increased by one when entered inside the τ argument. On one hand the inequality $p+q-1-j \geq p+q-1-k = p-1+\ell$ is satisfied; on the other hand all the indices $i_\ell, \ldots, i_1$ are $> k+1 \geq j+1$, so that the following relation is satisfied:

$$\partial_{j+2}^{p+q-1-j} \eta' \eta'' = \partial_{j+2}^{p-1+k-j} \eta''.$$

But we have also the relation:

$$\partial_{j+2}^{p-1+k-j}\eta_k \ldots \eta_1 b = \eta_j \ldots \eta_1 \partial_2^{p-1} b;$$

finally, the p-simplex b gives a 1-simplex $\partial_2^{p-1}b$, dimension 1, necessarily the η_0-degeneracy of the base-point, for the base space B is 1-reduced; the corresponding twist is trivial and the associated bottom perturbation is null. ∎

The previous demonstration is a little technical but more elementary than the original ones [10, 61] (see also [26]), where the interesting notion of *twisting cochain* is required and used to make more conceptual the result. The present demonstration is sufficient to give a *certificate* for the corresponding computer program.

8.4 The *effective* version of the Serre spectral sequence.

Let $F \hookrightarrow [E = F \times_\tau B] \to B$ be a simplicial fibration. The Serre spectral sequence gives a set of relations between the homology groups of F, E and B. In some particular cases, this spectral sequence gives a method allowing you to deduce the homology groups of one of the components, E for example, when the homology groups of the others (B and F) are given. An example of this sort has been given in Section 3.3.1 where $H_*(B)$ was deduced from $H_*(F)$ and $H_*(E)$, known.

But in the general case, the Serre spectral sequence *is not* an algorithm; see for example [42, pp 6 and 28] for a serious warning about this question, unfortunately not formalized: a *computational* environment is required there to obtain a *mathematical* statement of the obstacle. Section 3.3.2 was devoted to the first historical example where the spectral sequence method *failed* to compute a sphere homotopy group.

We show here the *effective* homology methods give very easily a *constructive* version of the Serre spectral sequence. For example the Kenzo program "stupidly" computes in one minute $\pi_6 S^3 = \mathbb{Z}_{12}$.

Theorem 132 — *An algorithm computes:*

$$(F_{EH}, B_{EH}, \tau) \mapsto E_{EH}$$

where:

1. $F_{EH} = (F, C_*(F), EC_*^F, \varepsilon_F)$ *is a version* with effective homology *of the fibre space* F*;*
2. $B_{EH} = (B, C_*(B), EC_*^B, \varepsilon_B)$ *is a version* with effective homology *of the base space* B*; we assume the base space* B *is* 1-reduced*: only one vertex, no non-degenerate 1-simplex;*
3. $\tau : B \to G$ *is a twisting operator with values in a simplicial group* G *acting over the fibre space* F*, defining the* twisted product $E = F \times_\tau B$*;*
4. $E_{EH} = (E, C_*(E), EC_*^E, \varepsilon_E)$ *is a version* with effective homology *of the total space* E*.*

In other words, you can *compute* the homology groups of the total space E, no mysterious unreachable differential, no extension problem at abutment; see [42, pp 6 and 28]. More important, if the total space E is one of the elements of a new "reasonable" construction, the object E_{EH} can again be used to obtain a version with effective homology of the new constructed object, and so on.

PROOF. We must construct the equivalence $\varepsilon_E : C_*(E) \lll\!\!\!\Longrightarrow\!\!\!\ggg EC_*^E$. It is obtained as the composition of two equivalences, $\varepsilon_E := \varepsilon' \circ \varepsilon''$, see Proposition 124 for the construction of such a composition.

The first equivalence ε' is produced by the twisted Eilenberg-Zilber Theorem:

$$\varepsilon' = \{C_*(F \times_\tau B) \overset{=}{\lll} C_*(F \times_\tau B) \ggg C_*(F) \otimes_t C_*(B)\}$$

where the left reduction is trivial. When ε' and in particular the *twisted* tensor product $C_*(F) \otimes_t C_*(B)$ are constructed, then we can construct the second necessary homotopy equivalence ε'', by applying the basic perturbation lemma to the difference between $C_*(F) \otimes_t C_*(B)$ and $C_*(F) \otimes C_*(B)$. Two equivalences are available:

$$\begin{aligned} \varepsilon_F &= \{C_*(F) \lll \widehat{C}_*^F \ggg EC_*^F\} \\ \varepsilon_B &= \{C_*(B) \lll \widehat{C}_*^B \ggg EC_*^B\} \end{aligned}$$

and we can construct their (non-twisted) tensor product (Proposition 60):

$$\varepsilon_{FB} = \{C_*(F) \otimes C_*(B) \lll \widehat{C}_*^F \otimes \widehat{C}_*^B \ggg EC_*^F \otimes EC_*^B.\}$$

A *filtration degree* is defined on the three tensor products according to the degree with respect the second factor $C_*(B)$, $\widehat{C}_B$ or EC_B. Let us introduce on the bottom left-hand chain-complex of this homotopy equivalence the necessary perturbation to obtain the twisted tensor product $C_*(F) \otimes_t C_*(B)$; the base space B is 1-reduced and according to Proposition 131, this perturbation decreases the filtration degree at least by 2.

The left reduction of ε_{FB} describes the left hand chain-complex $C_*(F) \otimes C_*(B)$ as a subcomplex of the top chain-complex $\widehat{C}_F \otimes \widehat{C}_B$, and we can apply the easy perturbation lemma to the left reduction; the perturbation can be so transferred to the top chain-complex $\widehat{C}_*^F \otimes \widehat{C}_*^B$, obtaining the same graded module with another differential $\widehat{C}_*^F \otimes_{t'} \widehat{C}_*^B$ with the same property (Proposition 131) about the filtration degree for the difference between the new and the old differential: this perturbation is nothing but a *copy* of the starting perturbation on a subcomplex of $\widehat{C}_*^F \otimes \widehat{C}_*^B$. The perturbation over $\widehat{C}_*^F \otimes \widehat{C}_*^B$ decreases the filtration de gree at least by 2; the homotopical component of the right reduction of ε'' increases the filtration degree at most by one; the nilpotency hypothesis is satisfied. The basic perturbation lemma can therefore be applied to the right reduction and the perturbation obtained for the top chain-complex and the equivalence ε'' is obtained.

Both components EC_*^F and EC_*^B are *effective* chain-complexes; their tensor product, whatever is the differential, is effective too. We have obtained a version *with effective homology* of the total space E. ■.

9 The Eilenberg-Moore spectral sequence.

9.1 Introduction.

Let $F \hookrightarrow [E = F \times_\tau B] \to B$ be a fibration; the *total space* E is a twisted product of the *base space* B by the *fibre space* F, the *twist* being defined by an appropriate *twisting operator* $\tau : B \to G$, see Definition 120 which in particular explains the role of the *structural group* G. As usual in constructive topology, we are working inside the *simplicial* framework.

The Serre spectral sequence, or more exactly the *effective homology* version of the Serre spectral sequence, see the previous section, allows us to compute the effective homology of the total space when the effective homologies of the base space and the fibre space are given; it is essentially a *product* operator. This is valid only if the base space is simply connected, more exactly in our simplicial framework, if the base space is 1-reduced.

The Eilenberg-Moore spectral sequence corresponds to a *division*. Because the notion of twisted product is not symmetric with respect to both factors, in fact *two* Eilenberg-Moore spectral sequences are to be defined, but they are similar. We will explain the *Cotor* spectral sequence, expressing the homology of the fibre space F as a "Cotor" operation between the homologies of the base space and the total space. The symmetric Tor spectral sequence describes the homology of the base space as a "Tor" involving the homologies of the total space, the fibre space *and the structural group*. We give here a reasonable level of details for the Cotor spectral sequence and will briefly explain how the symmetric result for the Tor spectral sequence is obtained.

9.2 Coalgebra and comodule structures.

The notions of algebra and module are common. The Cotor spectral sequence needs the symmetric notions of coalgebra and comodule.

Definition 133 — A *differential coalgebra* is a chain-complex C_* provided with a coproduct $\Delta : C_* \to C_* \otimes C_*$ and a counit $\eta : C_0 \to \mathfrak{R}$, satisfying the rules that are required for differential algebras, with the "arrows reversed".

See for example [36, VI.9]. In particular the tensor product $C_* \otimes C_*$ is itself a chain-complex (Definition 5.4) and the coproduct Δ must be *compatible with the differentials* of C_* and $C_* \otimes C_*$; in other words the coproduct is a *chain-complex morphism*. The coproduct is *homogeneous*: the (total) degree of a coproduct is equal to the degree of the argument: $|\Delta(x)| = |x|$. The coproduct is coassociative: $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$. The counit satisfies $(\eta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \eta) \circ \Delta = \mathrm{id}$; in these equalities, you have to identify $C_* \otimes \mathfrak{R} = \mathfrak{R} \otimes C_* = C_*$. All the tensor products, unless otherwise stated, are $\otimes_{\mathfrak{R}}$.

Definition 134 — Let X be a simplicial set. The *canonical coalgebra structure* of $C_*(X)$ is defined by the coproduct $\Delta(\sigma) = \sum_{i=0}^{n} \partial_{i+1} \cdots \partial_n \sigma \otimes \partial_0^i \sigma$ and the counit η is defined by $\eta(\sigma) = 1_{\mathfrak{R}}$ if $\sigma \in X_0$, $\eta(\sigma) = 0$ otherwise.

The formulas that are given for individual simplices must as usual be linearly extended to combinations of simplices. The coproduct is easily understood in the simplicial complex case:

$$\Delta(0123) = 0 \otimes 0123 + 01 \otimes 123 + 012 \otimes 23 + 0123 \otimes 3$$

where for example 123 denotes the simplex spanned by the vertices 1, 2 and 3. And there is a unique way to extend this game rule to the general case of simplicial sets. This coproduct is known as the *Alexander-Whitney* coproduct. The counit consists in deciding the image of a 0-simplex is the unit of the ground ring, and is null in higher dimensions. Usual algebraic topology uses this *coproduct* to install by duality a *product* on the cohomology, providing to this cohomology an *algebra structure*.

Definition 135 — If C_* is a differential coalgebra, a *differential* (right) *comodule* is a chain-complex M_* provided with an external coproduct $\Delta_M : M \to M \otimes C_*$, satisfying the rules that are required for differential algebras, with the "arrows reversed".

The external coassociativity rule becomes $(\Delta_M \otimes \mathrm{id}_C) \circ \Delta_M = (\mathrm{id}_M \otimes \Delta_C) \circ \Delta_M$. The external counit rule is $(\mathrm{id}_M \otimes \eta) \circ \Delta_M = \mathrm{id}_M$ where an identification $M \otimes \mathfrak{R} = M$ is necessary.

For example, if $f : X \to Y$ is a simplicial morphism between two simplicial sets, there is a canonical way to provide $C_*(X)$ with a $C_*(Y)$-comodule structure. Decide $\Delta_{X,Y} = (\mathrm{id}_{C_*X} \otimes f) \circ \Delta_X$: it is a process of coextension of scalars, the coproduct Δ_X of the *coalgebra* $C_*(X)$ is "extended" to a comodule coproduct $\Delta_{X,Y}$.

9.3 The Cobar construction.

If you intend to make *divisions*, a good idea could consist in firstly studying *inverses*: a division is most often nothing but a multiplication by an inverse. In topology, when you want to consider in a fibration $F \hookrightarrow E \to B$ the fibre space F as a (twisted) quotient of E by B, it is natural to look for some "inverse" of B.

Definition 136 — Let B be a *pointed* topologial space. The *path space* PB of B is the space of all the continuous maps $PB = \mathcal{C}((I,0),(B,*))$. The loop space ΩB of B is the space of all the continuous maps $\Omega B = \mathcal{C}((I,0,1),(B,*,*))$. A canonical fibration $\Omega B \hookrightarrow PB \to B$ is defined.

The space B is *pointed*, that is, B is a shorthand for $B = (B,*)$ where $*$ is some distinguished point of B, its *base point*. In the case of a path $\gamma \in PB$, the image of $0 \in I = [0, 1]$ must be the base point $*$. The same for 0 and 1 in the case of a *loop*. A loop is a path, but in general a path is not a loop. The canonical projection $\mathrm{pr} : PB \to B$ is defined by $\mathrm{pr}(\gamma) = \gamma(1)$. The fibre *above the base point* is the loop space ΩB; it is a Hurewicz fibration: the canonical projection $\mathrm{pr} : PB \to B$ satisfies the *homotopy lifting* property, see [62, 2.2]. The total space PB is contractible: every path can be retracted along itself to the trivial path $\gamma_0 \equiv *$. So that the total space has the homotopy type of a point, the unit in the world of topology. And the loop space ΩB is therefore a sort of *inverse* $\Omega B =$ "B^{-1}".

These constructions were invented by Jean-Pierre Serre when he designed appropriate tools allowing him to "compute" homotopy groups of spheres. But we cannot work with *general* topology on a computer, and the analogous process in *combinatorial* topology was discovered (or invented?) by Daniel Kan and was sketched Section 7.5.6. We summarize the corresponding result in this theorem.

Theorem 137 — *A functor Ω can be defined on the category of reduced simplicial sets. If B is such a simplicial set, then a canonical twisting operator τ is defined $\tau : B \to \Omega B$ defining a twisting product $PB = \Omega B \times_\tau B$ which is contractible.*

It is the simplicial version of the Hurewicz fibration $\Omega B \hookrightarrow PB \to B$. The chapter VI of [40] gives all the possible details about this question. In this way we have a simple process to construct the "inverse" of a *base* space.

The next step must go from combinatorial topology to algebra. It happens in a sense a differential coalgebra is the translation in algebra of a topological space[27].

UOStated 138 — *Our differential coalgebras C_* are assumed from now on* 1-reduced. *This means the 0-component C_0 is isomorphic to the ground ring $\mathfrak{R}$ by the coaugmentation η of the coalgebra, and the 1-component C_1 is null.*

Many definitions and results given here can be extended to significantly more general situations, but our main result is concerned by the 1-reduced case, and stating now this restriction makes easier the exposition.

Definition 139 — Let C be a differential coalgebra, and M (resp. N) be a right (resp. left) C-comodule. The *Cobar construction* $\mathrm{Cobar}^C(M, N)$ is a *bicomplex* defined as follows:

$$\mathrm{Cobar}^C(M, N) = \oplus_{p=0}^{\infty} (M \otimes \overline{C}^{\otimes p} \otimes N)^{[-p]}$$

where $\overline{C}$ is the *coaugmentation ideal* of C; the differential structure of $\mathrm{Cobar}^C(M, N)$ comes from two differentials, the vertical differential d_v is deduced from the component differentials, and the horizontal differential d_h is deduced from the various coproducts.

[27]A deeper study shows this is not exact; some essential information in general is *lost* in this translation process. If you want to keep the *whole* homotopy type of the space B, you must endow the chain complex $C_*(B)$ not only with the coalgebra structure, but with some E_∞-coalgebra structure, E_∞ being an appropriate algebraic *operad*, which can be understood as the completion of the Steenrod operations. See [7, 37] for this essential point.

It is a first quadrant bicomplex, the *horizontal* degree is p and the *vertical* degree is deduced from the grading of $(M \otimes \overline{C}^p \otimes N)$, each factor being graded. The total degree is the *difference* between vertical and horizontal degree, this is necessary because of the nature of the coproduct which *unfolds* an element into a sum of tensor products. This point is recalled by the exponent $[-p]$ in the initial formula; you can consider this difference as a *desuspension* process. A purist usually prefers install this bicomplex in the second quadrant, but the notation becomes a little heavier.

The reader notes we allow us not to indicate the grading property by the usual $*$-index, in order to trim the notation when it is possible. Let us detail a little more this definition of the Cobar construction. The coaugmentation $\eta : C \to \mathfrak{R}$ has a kernel $\overline{C}$; because of the restriction 138, the coaugmentation ideal $\overline{C}$ is nothing but C with the 0-component cancelled. The grading of $\overline{C}$ therefore begins in degree 2. The differential of $\overline{C}$ in degree 2 is null as in C itself, because of the absence of a 1-component.

The components M, $\overline{C}$ and N in the formula defining the Cobar are chain-complexes, so that their tensor products are chain-complexes too, see Definition 5.4; so is defined the *vertical* differential of the chain complex, signs being deduced from the Koszul rule, except the role of $(-1)^n$ explained later:

$$\begin{aligned} d_v(a \otimes c_1 \otimes \cdots c_n \otimes b) \quad = \quad & (-1)^n da \otimes c_1 \otimes \cdots \otimes c_n \otimes b \\ & + (-1)^{n+|a|} a \otimes dc_1 \otimes \cdots \otimes c_n \otimes b \\ & + \cdots \quad \cdots \\ & + (-1)^{n+|ac_1 \cdots c_{n-1}|} a \otimes c_1 \cdots \otimes dc_n \otimes b \\ & + (-1)^{n+|ac_1 \cdots c_n|} a \otimes c_1 \cdots \otimes c_n \otimes db \end{aligned}$$

The coalgebra C and the comodules M and N are provided with coproducts. The ideal $\overline{C}$ inherits a pseudo-coproduct again denoted by $\Delta : \overline{C} \to \overline{C} \otimes \overline{C}$ by cancelling in the original coproduct the factors of degree 0 in the result. For example in the case of the standard s-simplex Δ^s with the 1-skeletton collapsed on the base point to satisfy the 1-reduced requirement, we would have: $\Delta(0123) = 0$ because the 0- and 1-simplices do not exist anymore in $\overline{C}_*^N(\Delta^n)$; on the contrary $\Delta(01234) = 012 \otimes 234$. The same process for M and N gives pseudo-coproducts $\Delta : M \to M \otimes \overline{C}$ and $\Delta : N \to \overline{C} \otimes N$. Then the horizontal differential is defined, if $m \in M$, $c_i \in \overline{C}$ and $b \in N$, by the formula:

$$\begin{aligned} d_h(a \otimes c_1 \otimes \cdots \otimes c_n \otimes b) \quad = \quad & \Delta(a) \otimes c_1 \otimes \cdots \otimes c_n \otimes b \\ & - a \otimes \Delta(c_1) \otimes \cdots \otimes c_n \otimes b \\ & \pm \cdots \quad \cdots \\ & + (-1)^n a \otimes c_1 \otimes \cdots \otimes \Delta(c_n) \otimes b \\ & + (-1)^{n+1} a \otimes c_1 \otimes \cdots \otimes c_n \otimes \Delta(b). \end{aligned}$$

The Cobar carries in particular a *cosimplicial* structure. The p-simplices are the elements of $M \otimes \overline{C}^{\otimes p} \otimes N$ and the coface operator $\partial_i : M \otimes \overline{C}^{\otimes p} \otimes N \to M \otimes \overline{C}^{\otimes(p+1)} \otimes N$ is defined by applying the coproduct to the i-th copy of $\overline{C}$, or to M (resp. N) if $i = 0$ (resp. $i = p + 1$). As for the simplicial structures, a cosimplicial structure is defined by a *covariant* functor to the considered category, here the $\mathfrak{R}$-modules. The compatibility of the coproduct with the differentials guarantees every ∂_i is also compatible with the differentials. As in the simplicial case, the alternate sum of the coface operators is a differential[28], our horizontal differential d_h.

In this way every component of the horizontal differential $d_h : M \otimes \overline{C}^{\otimes p} \otimes N \to M \otimes \overline{C}^{\otimes (p+1)} \otimes N$ is a chain-complex morphism; as usual to obtain a bicomplex, we must transform the commutative squares into anticommutative squares, this is the role of the factor $(-1)^n$ in the formula for the vertical differential, which can also be considered as an effect of the desuspension process.

Many authors prefer to call d_v the *tensorial* differential and d_h the *cosimplicial* differential. Our terminology, vertical and horizontal refers to the bicomplex structure for our Cobar, which is very important in effective homology.

Theorem 140 — *A general algorithm computes:*

$$(M_{EH}, C_{EH}, N_{EH}) \mapsto \mathit{Cobar}^C(M,N)_{EH}$$

where:

1. *C_{EH} is a 1-reduced differential coalgebra with effective homology;*
2. *M_{EH} (resp. N_{EH}) is a right (resp. left) C-comodule with effective homolohy.*
3. *The result $\mathit{Cobar}^C(M,N)_{EH}$ is a version* with effective homology *of the Cobar construction $\mathit{Cobar}^C(M,N)$.*

Proof. It is a simple application of the Bicomplex Reduction Theorem 75. As usual, let us use the notation $C \lll \widehat{C} \ggg EC$ for the given equivalence between C and some effective chain-complex EC, the same for M and N. In a first step, we cancel the horizontal differential of $\mathrm{Cobar}^C(M,N)$, which is nothing but replacing the C-coproduct by $\Delta_0(x) = 1 \otimes x + x \otimes 1$, the unit 1 being defined by the coaugmentation, and for example the M-coproduct by $\Delta_0(x) = x \otimes 1$. We so obtain a simplified $\mathrm{Cobar}^{C_0}(M_0, N_0)$ which is a banal direct sum of tensor products and as a simple consequence of Proposition 60, we obtain an equivalence:

$$\mathrm{Cobar}^{C_0}(M_0,N_0) \lll \mathrm{Cobar}^{\widehat{C}_0}(\widehat{M}_0,\widehat{N}_0) \ggg \mathrm{Cobar}^{EC_0}(EM_0,EN_0)$$

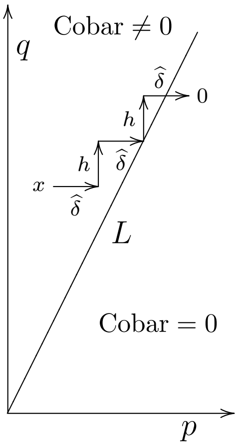
where the 0-index signals the coproduct is made or defined as trivial.

Now we reinstall into the initial Cobar the horizontal differential; this is a perturbation. For the left hand member of the new equivalence to be constructed, the so-called Easy Perturbation lemma 49 must be firstly applied. We obtain:

$$\mathrm{Cobar}^C(M,N) \lll \widetilde{\mathrm{Cobar}}^{\widehat{C}}(\widehat{M},\widehat{N}) \,\rule{2pt}{1em}\!\ggg \mathrm{Cobar}^{EC_0}(EM_0,EN_0)$$

[28]More precisely, the cosimplicial structure should be installed on $\oplus_{p=0}^{\infty}(M \otimes C^{\otimes p} \otimes N)^{[-p]}$; our horizontal differential is the differential obtained for the *normalized* chain-complex, the normalization consisting in this case in replacing every occurence of C by $\overline{C}$, see [36, X.2.2].

The tilde above the $\widetilde{\mathrm{Cobar}}$ explains there is some similarity between the new differential installed on the central object and a Cobar structure, but it is not actually a Cobar construction[29]. The vertical bar **|** signals the right hand reduction is no longer valid, because of the central perturbation.



For the right hand part of the equivalence, we must apply the actual BPL and there remains to verify the nilpotency condition. It is here that the 1-reduced property is required for our coalgebra C; because of this property, the grading in column p begins at the ordinate $2p$: the bicomplex is null under a line L of slope 2. The path which is followed when iterating the composition $h\widehat{\delta}$ (notations of Theorem 50) is a stairs of "slope" 1 which eventually goes under the line L, into an area where the Cobar bicomplex is null: the nilpotency hypothesis is satisfied.

The basic perturbation lemma produces a reduction between $\widetilde{\mathrm{Cobar}}^{\widehat{C}}(\widehat{M}, \widehat{N})$ and some pseudo-Cobar to be denoted as $\widetilde{\mathrm{Cobar}}^{EC}(EM, EN)$. We finally have an eauivalence:

$$\mathrm{Cobar}^{C}(M, N) \Lleftarrow \widetilde{\mathrm{Cobar}}^{\widehat{C}}(\widehat{M}, \widehat{N}) \Rrightarrow \widetilde{\mathrm{Cobar}}^{EC}(EM, EN)$$

Again because of the 1-*reduced* hypothesis for the coalgebra C, the right hand chain complex is *effective*: the total degree is defined as $q - p$ and a homogeneous component of this chain complex is made of pieces installed on a line of slope 1; but the intersection of this line with the triangle "Cobar $\neq 0$" is finite, and the corresponding homogeneous component therefore is *effective*. ■

It must be noted the perturbation lemma transforms $\mathrm{Cobar}^{EC_0}(EM_0, EN_0)$, a bicomplex, in fact without any horizontal differential, into $\widetilde{\mathrm{Cobar}}^{EC}(EM, EN)$, a multicomplex with arrows $d^r_{p,q}$, maybe non-null for arbitrary values of r; see Definition 74 and Theorem 75. The extra arrows so defined are at the origin of the notion of A_∞-coalgebra [63].

A particular case of the Cobar contraction is crucial.

Theorem 141 — *The Cobar construction* $\mathrm{Cobar}^C(C, \mathfrak{R})$ *is a coresolution of* $\mathfrak{R}$.

The prefix 'co' in coresolution means it is an injective resolution $\mathfrak{R} \to \mathrm{Cobar}^C(C, \mathfrak{R})$ instead of a projective resolution $\mathfrak{R} \leftarrow \mathrm{Cobar}^C(C, \mathfrak{R})$

We recall C is assumed reduced, so that C_0 is isomorphic to the ground ring $\mathfrak{R}$, which induces a left C-comodule structure $\mathfrak{R} \to C_0 \subset C = C \otimes \mathfrak{R}$.

Proof. The contraction h of the complex $\mathrm{Cone}(\mathfrak{R} \to \mathrm{Cobar}^C(C, \mathfrak{R}))$ is defined by:

$$h(c_0 \otimes c_1 \otimes \cdots \otimes c_n \otimes 1_{\mathfrak{R}}) = \eta(c_0) c_1 \otimes c_2 \otimes c_n \otimes 1_{\mathfrak{R}}$$

[29]The notion of A_∞-structure is designed to handle such a situation [63].

The verification is a simple calculation. In particular $\eta(c_0) \neq 0$ only if $c_0 \in C_0 \subset C$. ■

9.4 The *effective* Eilenberg-Moore spectral sequence.

In this section, a simplicial fibration $F \hookrightarrow [E = F \times_\tau B] \to B$ is given; in particular a group action $G \times F \to F$ for some simplicial group G and a twisting function $\tau : B \to G$ are present. In the Kenzo implementation, the twisting function τ <u>*is*</u> the fibration, which is the right point of view. Two equivalences $C_*(E) \twoheadleftarrow\!\!\!\twoheadleftarrow \widehat{E}_* \twoheadrightarrow\!\!\!\twoheadrightarrow EE_*$ and $C_*(B) \twoheadleftarrow\!\!\!\twoheadleftarrow \widehat{B}_* \twoheadrightarrow\!\!\!\twoheadrightarrow EB_*$ between the chain-complexes of E and B and *effective* chain-complexes EE_* and EB_* are given too, describing the total space E and the base space B as simplicial sets *with effective homology*. Note no homological information is required for the structural group G which can be any kind of *locally effective* simplicial group. The simplicial base space B *is 1-reduced*: a unique vertex, the base point, and no non-degenerate 1-simplex; the coalgebra $C_*(B)$ is also 1-reduced and the coaugmentation ideal $\overline{C_*(B)}$ begins only in degree 2.

Theorem 142 — *A general algorithm computes:*

$$[F, G, B_{EH}, \tau, E_{EH}] \mapsto F_{EH}$$

where

1. *F, G, B, τ and E are as explained above;*
2. *E_{EH} (resp. B_{EH}, F_{EH}) is a version with effective homology of E (resp. B, F).*

In other words, if the effective homology of the total space and of the base space are known, an algorithm computes the effective homology of the fibre space. Victor Gugenheim [27] computed an effective chain-complex, the homology of which is guaranteed being the homology of the fibre space, but this is not enough to iterate the process: an equivalence between the effective chain-complex and the chain-complex of the fibre space is then necessary; it is the key point to obtain a solution of the *Adams' problem.*

The next proposition which has its own interest will be used.

Proposition 143 — *In the data of the Basic Perturbation Lemma 50, if the relation $\widehat{\delta} g = 0$ holds, then the resulting perturbation δ for the small chain-complex is null: $\delta = 0$. The same if $f\widehat{\delta} = 0$.*

It is a paradoxical result. Usually the BPL is used to construct a *new* interesting differential for the small graded module C_*. It happens sometimes we are interested by two *different* reductions from the big graded module $\widehat{C}_*$ provided with two *different* differentials over the *same* small chain-complex C_*.

Proof. Just glance at the formula $\delta = f\widehat{\delta}\phi g = f\psi\widehat{\delta} g$ page 49. ■

Proof of Theorem 142. The Eilenberg-Zilber Theorem 123 produces a reduction $C_*(F \times B) \Rrightarrow C_*(F) \otimes C_*(B)$ and the twisted Eilenberg-Zilber Theorem 130 another reduction $C_*(F \times_\tau B) \Rrightarrow C_*(F) \otimes_t C_*(B)$. Theorem 141 produces a reduction $\mathfrak{R} \Lleftarrow \mathrm{Cobar}^{C_*(B)}(C_*(B), \mathfrak{R})$; applying to the last reduction the functor $C_*(F) \otimes {<}?{>}$ gives again another reduction:

$$(f, g, h) : C_*(F) \Lleftarrow \mathrm{Cobar}^{C_*(B)}(C_*(F) \otimes C_*(B), \mathfrak{R}).$$

Note in the Cobar the $C_*(B)$-comodule structure of $C_*(F) \otimes C_*(B)$ is induced by the canonical projection $C_*(F) \otimes C_*(B) \to C_*(B)$, which projection requires the coaugmentation $C_*(F) \to \mathfrak{R}$. The same for the twisted tensor product below.

We intend to replace in the last reduction the ordinary tensor product $C_*(F) \otimes C_*(B)$ by the twisted one $C_*(F) \otimes_t C_*(B)$; this is a perturbation $\widehat{\delta}$ of the Cobar differential. Is the condition $\widehat{\delta} g = 0$, which would allow us to apply Proposition 143, satisfied? Let us call $*_B$ the base point of B. The component g of the reduction maps $C_*(F)$ onto the *sub-chain-complex* $C_*(F) \otimes C_*(*_B)$ inside the 0-column of the Cobar bicomplex; this a subcomplex not only inside the 0-column, but also in the Cobar: the pseudo-coproduct $C_*(*_B) \to C_*(*_B) \otimes \overline{C_*(B)}$ is null, because of the restriction to the coaugmentation ideal which cancels the 0-component. This sub-chain-complex is left unchanged by the perturbation, for the *base* fibre of the total space is not modified by the twisting process. No perturbation on the sub-chain-complex $C_*(F) \otimes C_*(*_B)$ and the condition $\widehat{\delta} g$ holds. Proposition 143 produces a reduction:

$$(f', g, h') : C_*(F) \Lleftarrow \mathrm{Cobar}^{C_*(B)}(C_*(F) \otimes_t C_*(B), \mathfrak{R}). \tag{672}$$

because the g-component is also unchanged.

There remains to apply Theorem 140. The base space B is given with effective homology and the same for E; in other words an equivalence:

$$C_*(F \times_\tau B) \Lleftarrow \widehat{E}_* \Rrightarrow EE_*$$

is given. Composing the left hand reduction with the twisted Eilenberg-Zilber reduction $C_*(F) \otimes_t C_*(B) \Lleftarrow C_*(F \times_\tau B)$ gives an equivalence:

$$C_*(F) \otimes_t C_*(B) \Lleftarrow \widehat{E}_* \Rrightarrow EE_*,$$

in other words the chain complex $C_*(F) \otimes_t C_*(B)$ is with effective homology. Theorem 140 can be applied which produces an equivalence:

$$\mathrm{Cobar}^{C_*(B)}(C_*(F) \otimes_t C_*(B), \mathfrak{R}) \Lleftarrow \Rrightarrow \widetilde{\mathrm{Cobar}}^{EB_*}(EE_*, \mathfrak{R}).$$

Finally composing the left hand reduction with the reduction (672) above gives an equivalence:

$$C_*(F) \Lleftarrow\!\!\Rrightarrow \widetilde{\mathrm{Cobar}}^{EB_*}(EE_*, \mathfrak{R})$$

where the right hand chain-complex is effective. ∎

9.5 Adams' problem.

Our effective version of the Eilenberg-Moore spectral sequence gives a very simple solution to Adams' problem. Frank Adams ([1], see also [2]) designed an algorithm computing the homology groups of the *first* loop space ΩX of a 1-reduced simplicial set; stated in our framework, Adams'result is the following.

Theorem 144 (Adams' Theorem) — *Let X be a 1-reduced simplicial set. Then there exists a canonical isomorphism between $H_*(\Omega X; \mathfrak{R})$ and $H_*(\mathit{Cobar}^{C_*(B)}(\mathfrak{R}, \mathfrak{R}))$.*

If X is a finite 1-reduced simplicial set, the Cobar is effective and the homology groups are computable. Adams then asked for some analogous solution for the *iterated* loop space $\Omega^n X$. Eighteen (!) years later, Hans Baues [4] gave a solution for the second loop space $\Omega^2 X$; it depends on an ingenious possible geometrical model for the second loop space; but again it is not possible to extend this model to the third loop space $\Omega^3(X)\dots$

The problem is in fact in the *non-constructive* nature of Adams' solution for the first loop space. Elementary homological algebra shows that for reasonable ground rings $\mathfrak{R}$ there *exists* an equivalence $C_*(\Omega X) \lll\!\!\Longleftrightarrow\!\!\ggg \mathrm{Cobar}^{C_*(B)}(\mathfrak{R}, \mathfrak{R})$, but the exact nature of this equivalence is not studied.

Our *effective* Eilenberg-Moore spectral sequence on the contrary will *constructively* prove the existence of this equivalence; and then the iteration of the process is obvious, giving our solution to Adams' problem. So simple that it is not difficult to implement it on a computer, leading to programs computing homology groups of loop spaces otherwise so far unreachable.

We must make more precise Theorem 137.

Theorem 145 — *Let B be a 1-reduced* locally effective *simplicial set. Then the path space PB defined in Theorem 137 has* effective *homology.*

Proof. See for example [40, Chapter VI]. An *explicit* contraction is there *constructed* for the chain-complex $C_*(PB) = C_*(\Omega B \times_\tau B)$ for the appropriate twist τ defining the path space. It is easy to organize this contraction as a reduction $C_*(\Omega B \times_\tau B) \Rrightarrow \mathfrak{R}$. ■

Corollary 146 (Effective Adams' Theorem) — *A general algorithm computes:*

$$B_{EH} \mapsto (\Omega B)_{EH}$$

where B_{EH} is a 1-reduced simplicial set with effective homology *(input) and $(\Omega B)_{EH}$ is a version* with effective homology *of the loop space (output).*

Proof. Apply Theorem 142 to the fibration: $\Omega B \hookrightarrow [E = \Omega B \times_\tau B] \to B$. The base space B is given with its effective homology and the effective homology of the total space $E = PB = \Omega B \times_\tau B$ is computed by Theorem 145. ■

Corollary 147 (Solution to Adams' problem) — *A general algorithm computes:*

$$(n, X_{EH}) \mapsto (\Omega^n X)_{EH}$$

where the input X_{EH} is an r-reduced simplicial set with effective homology, *$r \geq n$, and the output is a version* with effective homology *of the n-th loop space. In particular the* ordinary *homology of this iterated loop space is computable.*

The qualifier *r-reduced* for X means in the simplicial structure of X there is no non-degenerate simplex in dimension $\leq n$ except the base point in dimension 0.

Proof. The simplicial model ΩX for the r-reduced simplicial set X is itself $(r-1)$-reduced. It is sufficient to successively apply n times Corollary 146. ■

9.6 Other Eilenberg-Moore spectral sequences.

The reader can be puzzled by the non-symmetric presence of the ground ring $\mathfrak{R}$ in $\mathrm{Cobar}^{C_*(B)}(E, \mathfrak{R})$, the main ingredient in the Eilenberg-Moore process. In fact our presentation is a particular case of a more general situation.

Definition 148 — Let $F \hookrightarrow [E = F \times_\tau B] \to B$ be a fibration and $\beta : B' \to B$ be a simplicial map. These data define an *induced fibration* $F \hookrightarrow E' \to B'$.

The twisting function τ is some "degree" -1 map $\tau : B \to F$, see Definition 120. The composition $\tau' = \tau\beta$ also is a twisting function, defining the induced fibration. Another point of view consists in thinking of the total space E' as the cartesian product $E' = B' \times_B E$, where the set of n-simplices E'_n is $E'_n = \{(\sigma', \sigma) \in B'_n \times E_n \ \textbf{st} \ f(\sigma') = \mathrm{pr}(\sigma)\}$ if pr is the projection $\mathrm{pr} : E \to B$. Both definitions are elementarily equivalent.

Theorem 149 (First effective Eilenberg-Moore spectral sequence) — *A general algorithm computes:*

$$(B_{EH}, F, G, \tau, E_{EH}, B'_{EH}, \beta) \mapsto E'_{EH}$$

where all the ingredients are as above, the EH-index meaning the corresponding object is given (case of B, E and B') or produced (case of E') with effective homology*.*

Theorem 142 is the particular case where β is the inclusion of the base point in the base space $*_B \hookrightarrow B$; the induced fibration is then simply $F \hookrightarrow F \to *_B$. The same method constructs in the general case an equivalence:

$$C_*(E') \lll\!\!\Longrightarrow\!\!\ggg \mathrm{Cobar}^{C_*(B)}(C_*(E), C_*(B')).$$

Note in particular β defines, even if the map $\beta : B' \to B$ is not a fibration, a $C_*(B)$-comodule structure on $C_*(B')$, which makes coherent the definition of the

Cobar. The proof is the same, you just have to replace the right hand $\mathfrak{R}$ in the various Cobars by $C_*(B')$.

What about the symmetric "division"? If $F \hookrightarrow E \rightarrow B$ is a fibration, we could also be interested by something like $B = E/F$ and we would like to deduce the effective homology of the base space B_{EH} from F_{EH} and E_{EH}; possible? Yes, it is the second Eilenberg-Moore spectral sequence. The general case works as follows. The main ingredients are two simplicial sets E and E' and a simplicial group G. A right (resp. left) action is given $\alpha : E \times G \rightarrow E$ (resp. $\alpha' : G \times E' \rightarrow E'$). This defines a cocartesian product $E \times_G E' := (E \times E')/ \sim_G$ where the equivalence relation $\sim_G$ makes equivalent $(\alpha(\sigma, \gamma), \sigma') \sim_G (\sigma, \alpha'(\gamma, \sigma'))$ when $\sigma \in E_n$, $\sigma' \in E'_n$ and $\gamma \in G$; think of the definition of a tensor product which, from a categorical point of view, is analogous.

In the first Eilenberg-Moore spectral sequence, there must be a fibration connecting the factor E of $B' \times_B E$ with the base space B. The second spectral sequence depends on an analogous requirement: one action, for example the first one $\alpha : E \times G \rightarrow E$ must define a principal fibration $G \hookrightarrow E \rightarrow E/G$ where E/G is nothing but $E \times_G *_{E'}$. If this condition is satisfied, an analogous *effective* spectral sequence is obtained.

Theorem 150 (Second *effective* Eilenberg-Moore spectral sequence) — *A general algorithm computes:*

$$(G_{EH}, E_{EH}, E'_{EH}, \alpha, \alpha') \mapsto (E \times_G E')_{EH}$$

where the ingredients are as above, the EH-index meaning the corresponding object is provided with effective homology*. The structural group G is assumed 0-reduced.*

The proof is the same, the key intermediate ingredient being the Bar construction $\mathrm{Bar}^{C_*(G)}(C_*(E), C_*(E'))$. In fact the various multiplicative structures define a structure of differential algebra over $C_*(G)$, sometimes called the Pontrjagin structure, and $C_*(G)$-module structures on $C_*(E)$ and $C_*(E')$. Note this time the effective homology of the structural group is also *required*: it plays the role of the base space B in the symmetric situation.

It has been explained the loop space ΩX can be considered as a twisted inverse of the original space X, for the appropriate twisted product $\Omega X \times_\tau X$ is contractible, has the homotopy type of a point, and the point is the unit in the topological world. In the same way, the classifying space construction [40, §21] allows one to construct the *universal fibration* $G \hookrightarrow EG \rightarrow BG$ where the total space EG is contractible and the base space is an "inverse" of G. In particular if G is the Eilenberg-MacLane space $K(\pi, 1)$, see Section 7.5.5, then $BG = K(\pi, 2)$ and more generally $B^{n-1}G = K(\pi, n)$. For sensible commutative groups π, Theorem 150 can compute the effective homology of $K(\pi, n)$. Coming back to the rough explanations given in Section 3.3.2 about the computation of homotopy groups, it is easy to prove:

Theorem 151 — *A general algorithm computes:*

$$(n, X_{EH}) \rightarrow \pi_n X$$

where X_{EH} is a 1-reduced simplicial set with effective homology and $\pi_n X$ is the n-th homotopy group of X.

This is a powerful generalization of Edgar Brown's Theorem [9]: the scope is much larger than in Edgar Brown's paper where the simplicial set is assumed *finite*, and the proof is more conceptual, so conceptual that the machine implementation is not very difficult. See the Kenzo documentation [19].

10 The claimed Postnikov "invariants"[30]

10.1 Introduction.

> *As yet we are ignorant of an effective method of computing the cohomology of a Postnikov complex from π_n and k^{n+1}* [23].

When this paper is written, the so-called *Postnikov invariants* (or k-invariants) are roughly fifty years old [47]; they are a key component of standard Algebraic Topology. This notion is so important that it is a little amazing to observe some important *gaps* are still present in our working environment around this subject, still more amazing to note these gaps are seldom considered. One of these "gaps" is unfortunately an *error*, widely spread, and easy to state: the terminology "Postnikov *invariants*" is incorrect: any sensible definition of the *invariant* notion leads to the following conclusion: the Postnikov invariants are not… invariants. This is true even in the simply connected case and to make easier the understanding, we restrict our study to this case.

First, several interesting questions of *computability* are arisen by the very notion of Postnikov invariant. It is surprisingly difficult fo find citations related to this computability problem, as though this problem was unconsciously "hidden" (?) by the topologists. The only significant one found by the authors is the EDM title quotation[31]. In fact there are *two* distinct problems of this sort.

On one hand, if a simply connected space is presented as a *machine object*, does there exist a general algorithm computing its Postnikov invariants? The authors have designed a general framework for *constructive* Algebraic Topology, giving in particular such a general algorithm [56, 52]. In the text, this process is formalized as a functor $\mathbf{SP} : \mathcal{SS}_{EH} \widetilde{\times} I \rightarrow \mathcal{P}$ where $\mathcal{SS}_{EH}$ is an appropriate category of *computable* topological spaces, and $\mathcal{P}$ is the *Postnikov category.* We will explain later the nature of the factor I, in fact the heart of our subject.

[30]This section is a rough copy of the paper [54]; which explains some redundancies and also some gaps with respect to the previous sections; in spite of these defects, we think a reader having reached this point of the text could be interested by this section, making obvious a surrealist error in the standard terminology of Algebraic Topology.

[31]Other possible quotations are welcome.

On the other hand, a *converse* problem must be considered. When a Postnikov tower is given, that is, a collection of homotopy groups and *relevant* Postnikov invariants, how to construct the corresponding topological space? The computability problem stated in the title quotation is a (small) part of this converse problem. Again, our notion of *constructive* Algebraic Topology entirely solves it. The resulting computer program Kenzo [19] allows us to give a simple concrete illustration. In fact it will be explained it is not possible to properly *state* this problem... without having a solution of it! Again a strange situation to our knowledge not yet considered by the topologists. Our solution for the converse problem will be formalized as a functor $\mathbf{PS} : \mathcal{P} \to \mathcal{SS}_{EH}$.

There is a lack of symmetry between the functors $\mathbf{SP} : \mathcal{SS}_{EH} \widetilde{\times} I \to \mathcal{P}$ and $\mathbf{PS} : \mathcal{P} \to \mathcal{SS}_{EH}$. Instead of our functor $\mathbf{SP} : \mathcal{SS}_{EH} \widetilde{\times} I \to \mathcal{P}$, a simpler functor $\mathbf{SP} : \mathcal{SS}_{EH} \to \mathcal{P}$, without the mysterious factor I, is expected, but in the current state of the art, such a functor *is not* available. It is a consequence of the following *open* problem: let $P_1, P_2 \in \mathcal{P}$ be two Postnikov towers; does there exist an algorithm deciding whether $\mathbf{PS}(P_1)$ and $\mathbf{PS}(P_2)$ have the same homotopy type or not? The remaining uncertainty is measured by the factor I. And because of this uncertainty, the so-called Postnikov invariants are not... invariants: the context clearly says they should be invariants of the *homotopy type*, but such a claim is equivalent to a solution of the above decision problem.

It is even possible this decision problem *does not have* any solution; in fact, our Postnikov decision problem can be translated into an arithmetical decision problem, a subproblem of the general tenth Hilbert problem to which Matiyasevich gave a negative answer [39]. If our decision problem had in turn a negative answer, it would be *definitively* impossible to transform the common Postnikov invariants into *actual* invariants.

10.2 The Postnikov category and the PS functor.

Defining a functor $\mathbf{PS} : \mathcal{P} \to \mathcal{SS}_{EH}$ in principle consists in defining the *source* category, here the *Postnikov category* $\mathcal{P}$, the *target* category, the *simplicial set category* $\mathcal{SS}_{EH}$, and *then*, finally, the functor $\mathbf{PS}$ itself. It happens this is not possible in this case : the Postnikov category $\mathcal{P}$ and the functor $\mathbf{PS}$ are *mutually recursive*. More precisely, an object $P \in \mathcal{P}$ is a limit $P = \lim P_n$, every P_n being also an element of $\mathcal{P}$. Let $\mathcal{P}_n$, $n \geq 1$, be the Postnikov towers limited to dimension n. The definition of $\mathcal{P}_{n+1}$ *needs* the partial functor $\mathbf{PS}_n : \mathcal{P}_n \to \mathcal{SS}_{EH}$ where $\mathbf{PS}_n = \mathbf{PS}|\mathcal{P}_n$ and this is why the definitions of $\mathcal{P}$ and $\mathbf{PS}$ are mutually recursive.

We work only with simply connected spaces, the homotopy (or $\mathbb{Z}$-homology) groups of which being of *finite type*. It is essential, when striving to define *invariants*, to have exactly *one* object for every isomorphism class of groups of this sort, so that we adopt the following definition. No p-adic objects in our environment, which allows us to denote $\mathbb{Z}/d\mathbb{Z}$ by $\mathbb{Z}_d$; in particular $\mathbb{Z}_0 = \mathbb{Z}$.

Definition 152 — A *canonical group* (abelian, of finite type) is a product $\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_k}$ where the non-negative integers d_i satisfy the divisibility condition: d_i divides d_{i+1} for $1 \leq i < k$.

Every abelian group of finite type is isomorphic to *exactly one* canonical group, for example the group $\mathbb{Z}^2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{15}$ is isomorphic to the unique canonical group $\mathbb{Z}_{30} \times \mathbb{Z}_{30} \times \mathbb{Z}_0 \times \mathbb{Z}_0$; but such an isomorphism is *not*... canonical; for example, for the previous example, there exists an infinite number of such isomorphisms, and we will see this is the key point preventing us from qualifying the Postnikov invariants as invariants.

Definition 153 — The category $\mathcal{SS}_{EH}$ is the category of the *simply connected simplicial sets with effective homology* described in [52].

The framework of the present paper does not allow us to give the relatively complex definition of this category. Roughly speaking, an object of this category is a *machine object* coding a (possibly infinite) simply connected simplicial set with *known* homology groups; furthermore a *complete* knowledge of the homology is required: mainly every homology class has a canonical representant cycle, an algorithm computes the homology class of every cycle, and if two cycles c_0 and c_1 are homologous, an algorithm computes a chain C with $\partial C = c_1 - c_0$. For example it is explained in [53] that $X = \Omega(\Omega(P^\infty(\mathbb{R})/P^3(\mathbb{R})) \cup_4 D^4) \cup_2 D^3$ is an object of $\mathcal{SS}_{EH}$ and the Kenzo program does compute the first homology groups of it, in the detailed form just briefly sketched. More generally every "sensible" simply connected space with homology groups of finite type has the homotopy type of an object of $\mathcal{SS}_{EH}$; this statement is precisely stated in [52], the proof is not hard, it is only a repeated application of the so-called *homological perturbation lemma* [11] and the *most detailed* proof is the Kenzo computer program itself [19], a Common Lisp text of about 16,000 lines.

The definitions of the category $\mathcal{P}$ and the functor $\mathbf{PS}$ are *mutually recursive* so that we need a starting point.

Definition 154 — The category $\mathcal{P}_1$ has a unique object, the void sequence $()_{2 \leq n \leq 1}$, the trivial Postnikov tower, and the functor $\mathbf{PS}_1$ associates to this unique object the trivial element $* \in \mathcal{SS}_{EH}$ with only a base point.

The next definitions of the category $\mathcal{P}_n$ and the functor $\mathbf{PS}_n$ assume the category $\mathcal{P}_{n-1}$ *and* the functor $\mathbf{PS}_{n-1} : \mathcal{P}_{n-1} \to \mathcal{SS}_{EH}$ are already available.

Definition 155 — An object $P_n \in \mathcal{P}_n$ is a sequence $((\pi_m, k_m))_{2 \leq m \leq n}$ where:

- $((\pi_m, k_m))_{2 \leq m \leq n-1}$ is an element $P_{n-1} \in \mathcal{P}_{n-1}$;
- The component π_n is a canonical group;
- The component k_n is a cohomology class $k_n \in H^{n+1}(\mathbf{PS}_{n-1}(P_{n-1}), \pi_n)$;

Let us denote $X_{n-1} = \mathbf{PS}_{n-1}(P_{n-1})$. The cohomology class k_n classifies a fibration :

$$K(\pi_n, n) \hookrightarrow K(\pi_n, n) \times_{k_n} X_{n-1} \twoheadrightarrow X_{n-1} \xrightarrow{k_n} K(\pi_n, n+1) = BK(\pi_n, n).$$

- Then the functor $\mathbf{PS}_n$ associates to $P_n = ((\pi_m, k_m))_{2\leq m\leq n} \in \mathcal{P}_n$ a version *with effective homology* $X_n = \mathbf{PS}_n(P_n)$ of the total space $K(\pi_n, n) \times_{k_n} X_{n-1}$.

In particular our version *with effective homology* of the Serre spectral sequence and our versions *with effective homology* of the Eilenberg-MacLane spaces $K(\pi, n)$ allow us to construct a version also with effective homology of the total space $K(\pi_n, n) \times_{k_n} X_{n-1}$, here denoted by X_n. We will give a typical small Kenzo demonstration at the end of this section.

A canonical forgetful functor $\mathcal{P}_n \to \mathcal{P}_{n-1}$ is defined by forgetting the last component of $((\pi_m, k_m))_{2\leq m\leq n}$, which allows us to define $\mathcal{P}$ as the projective limit $\mathcal{P} = \varprojlim \mathcal{P}_n$. If X_{n-1} is a simplicial set, the $(n-1)$-skeletons of X_{n-1} and $K(\pi_n, n) \times_{k_n} X_{n-1}$ are the same (for the standard model of $K(\pi_n, n)$), so that if $P = \varprojlim P_n$, the limit $\mathbf{PS}(P) = \varprojlim \mathbf{PS}_n(P_n)$ is defined also as an object of $\mathcal{SS}_{EH}$. The category $\mathcal{P}$ and the functor $\mathbf{PS} : \mathcal{P} \to \mathcal{SS}_{EH}$ are now *properly* defined.

The homotopy groups π_m's of a Postnikov tower $((\pi_m, k_m))_{2\leq m}$ can be defined *firstly* independently of the k_m's, but k_n can be *properly defined* only when $((\pi_m, k_m))_{2\leq m<n}$ is given *and only if* the functor $\mathbf{PS}_{n-1}$ is available in the environment. In other words, if the problem of the title EDM quotation is not solved, the very notion of Postnikov tower cannot be made *effective*.

10.3 Kenzo example.

Let us play the game consisting in constructing the beginning of a Postnikow tower with a $\pi_i = \mathbb{Z}_2$ at each stage and the "simplest" *non-trivial* Postnikov invariant. First $P_1 = ()$ and $X_1 = \mathbf{PS}_1(P_1) = *$. As planned, we choose $\pi_2 = \mathbb{Z}_2$ and $k_2 \in H^3(X_1, \mathbb{Z}_2) = 0$ is necessary null, no choice. So that we define $P_2 = ((\mathbb{Z}_2, 0))$ and $X_2 = K(\mathbb{Z}_2, 2)$. The Kenzo function `k-z2` can construct this space. We show a copy of the dialog between a Kenzo user and the Lisp machine.

```
> (setf X2 (k-z2 2)) ✠
[K13 Abelian-Simplicial-Group]
```

This dialog goes as follows. The Lisp *prompt* is the bigger sign '`>`'. The Lisp user enters a Lisp *statement*, here "`(setf X2 (k-z2 2))`". The Maltese cross '✠' signals the end of the statement to be executed, it is added here to help the reader, but it is not visible on the user screen. When the Lisp statement is finished, Lisp *evaluates* it, the computation time can be a microsecond or a few days or more, depending on the statement to be evaluated, and when the evaluation *terminates*, a Lisp object is *returned*, most often it is the "result" of the computation. Here the `K13` object (the Kenzo object #13) is constructed and returned, it is an abelian simplicial group. A Lisp statement "`(setf some-symbol (some-function some-arguments))`" orders Lisp to make the function `some-function` work, using the arguments `some-arguments`; this function creates some object which is *returned* (displayed) *and* assigned to the symbol `some-symbol`; in this way, the created object remains reachable through the symbol locating it.

The $\mathbb{Z}$-homology in dimensions 3 and 4 of X_2 (the arguments 3 and 5 must be understood as defining $3 \leq i < 5$):

```
> (homology X2 3 5) ✠
Homology in dimension 3 :
---done---
Homology in dimension 4 :
Component Z/4Z
---done---
```

to be read $H_3 = 0$ and $H_4 = \mathbb{Z}_4$. The universal coefficient theorem implies $H^4(X_2, \mathbb{Z}_2) = \mathbb{Z}_2$, there is only one non-trivial possible $k_3 \in H^4(X_2, \mathbb{Z}_2)$ and the Kenzo function `chml-clss` (<u>c</u>o<u>h</u>omology <u>cl</u>a<u>ss</u>) constructs it.

```
> (setf k3 (chml-clss X2 4)) ✠
[K125 Cohomology-Class on K30 of degree 4]
```

The attentive reader can be amazed to see this cohomology class defined on `K30` and not `K13` $= X_2$. The explanation is the following. Let us consider the *<u>ef</u>fective <u>h</u>o<u>m</u>ology* of X_2:

```
> (efhm X2) ✠
[K122 Equivalence K13 <= K112 => K30]
```

This is a chain equivalence between the chain complex of the considered space and some *small* chain complex, here the chain complex `K30`. In fact it is a *strong* chain equivalence, made of two *reductions* through the intermediate chain complex `K112` (see [52] for details). So that defining a cohomology class of X_2 is equivalent to defining such a class for `K30`. A *small* chain complex is a free $\mathbb{Z}$-chain complex of finite type in every dimension. The chain complex `K13` of the standard model of $X_2 = K(\mathbb{Z}_2, 2)$ is already of finite type, but the complex `K30` is much smaller. For example, in dimension 6, `K13` has 27,449 generators and `K30` has only 5.

The k_3 class allows us to define the fibration canonically associated:

$$F_3 = \left\{ K(\mathbb{Z}_2, 3) \hookrightarrow K(\mathbb{Z}_2, 3) \times_{k_3} X_2 \twoheadrightarrow X_2 \xrightarrow{k_3} K(\mathbb{Z}_2, 4) \right\}$$

We have now the Postnikov tower $P_3 = ((\mathbb{Z}_2, 0), (\mathbb{Z}_2, k_3))$ with $X_3 = \mathbf{PS}(P_3) = K(\mathbb{Z}_2, 3) \times_{k_3} X_2$. The Kenzo program can construct our fibration F_3 and its total space X_3.

```
> (setf F3 (z2-whitehead X2 k3)) ✠
[K140 Fibration K13 -> K126]
> (setf X3 (fibration-total F3)) ✠
[K146 Kan-Simplicial-Set]
```

The fibration is modelled as a *twisting operator* $\tau_3 : X_2 \to K(\mathbb{Z}_2, 3)$ which is nothing but an avatar of k_3, and we can verify the target of τ_3 is really $K(\mathbb{Z}_2, 3)$.

```
> (k-z2 3) ✠
[K126 Abelian-Simplicial-Group]
```

We continue to the next stage of our Postnikov tower. We “choose” again $\pi_4 = \mathbb{Z}_2$, but what about the next Postnikov invariant k_4? We must choose some $k_4 \in H^5(X_3, \mathbb{Z}_2)$, so that we are in front of the problem stated in the framed EDM title quotation. Fortunately, the Kenzo program *knows* how to compute the *necessary* H^5, the Kenzo program knows a (simple) solution for the EDM problem. In fact it knows the *effective* homology of the fibre space $K(\mathbb{Z}_2, 3)$:

```
> (efhm (k-z2 3)) ✠
[K268 Equivalence K126 <= K258 => K254]
```

In the same way, it knows the *effective* homology of $X_2 = K(\mathbb{Z}_2, 2)$, and the implicitly used *effective homology version* of the Serre spectral sequence, available in Kenzo, determines the effective homology of the twisted product X_3:

```
> (efhm X3) ✠
[K358 Equivalence K146 <= K348 => K344]
```

The chain-complex `K344` is of finite type, its homology groups are computable, and in this way Kenzo can compute the $\mathbb{Z}$-homology groups of X_3.

```
> (homology X3 2 6) ✠
Homology in dimension 2 :
Component Z/2Z
---done---
Homology in dimension 3 :
---done---
Homology in dimension 4 :
Component Z/2Z
---done---
Homology in dimension 5 :
Component Z/4Z
---done---
```

Finally the universal coefficient theorem implies $H^5(X^3, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and there are exactly four ways to add a new stage at our Postnikov tower with $\pi_4 = \mathbb{Z}_2$.

Four possible Postnikov *invariants* k_4. In this simple case, rather misleading, it is true such a k_4 is an *invariant* of the homotopy type of the resulting space, but in the general case, we will see the situation is much more complicated; this will be explained Section 10.6.2

The `chml-clss` Kenzo function constructs in such a case the cohomology-class "dual" to the generator of $H_5(X_3, \mathbb{Z}) = \mathbb{Z}_4$.

```
> (setf k4 (chml-clss X3 5)) ✠
[K359 Cohomology-Class on K344 of degree 5]
```

and the process can be iterated as before, giving the fibration F_4 associated to k_4, and the total space $X_4 = \mathbf{PS}_4(P_4) = K(\mathbb{Z}_2, 4) \times_{k_4} X_3$ with $P_4 = ((\mathbb{Z}_2, 0), (\mathbb{Z}_2, k_3), (\mathbb{Z}_2, k_4))$.

Constructing the next stage of the Postnikov tower *needs* the knowledge of $H^6(X_4, \mathbb{Z}_2)$, again a particular case of the EDM problem, and Kenzo computes in a few seconds $H^6(X_4, \mathbb{Z}_2) = \mathbb{Z}_2^4$: 16 different choices for the next Postnikov invariant k_5; again Kenzo knows how to directly construct the "simplest" non-trivial invariant k_5, in a sense which cannot be detailed here[32]; the other cohomology classes could be constructed and used as well, but the computations would be more complicated. Then F_5 and X_5 are constructed, but this time a few hours of computation are necessary to obtain $H^7(X_5, \mathbb{Z}_2) = \mathbb{Z}_2^5$: there are 32 different choices for the next invariant k_6 and again, in this "simple" case, such a k_6 actually is an invariant of the homotopy type of the resulting space, see Section 10.6.2.

And so on.

10.4 Morphisms between Postnikov towers.

10.4.1 The definition.

We have presented the Postnikov towers as being the objects of the Postnikov category $\mathcal{P}$, so that we must also describe the $\mathcal{P}$-morphisms. The standard considerations around homotopy groups and Kan minimal models, see for example [40], lead to the following definition.

Definition 156 — Let $P = ((\pi_n, k_n))_{n\geq 2}$ and $P' = ((\pi'_n, k'_n))_{n\geq 2}$ be two Postnikov towers. A *morphism* $f : P \to P'$ is a collection of group morphisms $f = (f_n : \pi_n \to \pi'_n)_{n\geq 2}$ satisfying the following *recursive* coherence property for every n. The sub-collection $(f_i)_{2\leq i\leq n-1}$, if coherent, defines a continuous map $\phi_{n-1} : X_{n-1}(= \mathbf{PS}(P_{n-1})) \to X'_{n-1}(= \mathbf{PS}(P'_{n-1}))$ between the $(n-1)$-th stages of the respective Postnikov towers. So that two canonical maps are defined:

- The map ϕ_{n-1} induces in a contravariant way a map $\phi^*_{n-1} : H^{n+1}(X'_{n-1}, \pi'_n) \to H^{n+1}(X_{n-1}, \pi'_n)$ between the cohomology groups;

[32]Depending on the Smith reduction of the boundary matrices of the small chain complex which is the main component of the effective homology of X_4.

- The map f_n induces in a covariant way a map $f_{n*} : H^{n+1}(X_{n-1}, \pi_n) \to H^{n+1}(X_{n-1}, \pi'_n)$.

Then the equality $\phi^*_{n-1}(k'_n) = f_{n*}(k_n)$ is required.

If so, a continuous map $\phi_n : X_n \to X'_n$ is defined, which allows one to continue the recursive process. The projective limit $\phi = \varprojlim \phi_n$ then is a continuous map $\phi : X = \mathbf{PS}(P) \to X' = \mathbf{PS}(P')$. $\square$

10.4.2 First example.

This definition implies some *isomorphisms* between *different* Postnikov towers can exist. Let us examine when a collection $f = (f_n : \pi_n \to \pi'_n)_{n\geq 2} : ((\pi_n, k_n))_{n\geq 2} \to ((\pi'_n, k'_n))_{n\geq 2}$ is an isomorphism. On one hand the coherence condition stated above must be satisfied, on the other hand every f_n must be a group isomorphism; if this is the case the obvious inverse $g = (f_n^{-1})_{n\geq 2}$ also satisfies the coherence condition and actually is an inverse of f.

The simplest example where a non-trivial isomorphism happens is the following. Let us consider the small Postnikov tower $P = ((\mathbb{Z}, 0), (\mathbb{Z}, k_3))$ where $k_3 \in H^4(K(\mathbb{Z}, 2))$ is $k_3 = c_1^2$, the square of the canonical generator $c_1 \in H^2(K(\mathbb{Z}, 2), \mathbb{Z})$, the first universal Chern class. The corresponding space $X = \mathbf{PS}(P)$ is the total space of a well defined fibration:

$$K(\mathbb{Z}, 3) \hookrightarrow X \twoheadrightarrow K(\mathbb{Z}, 2) \xrightarrow{c_1^2} K(\mathbb{Z}, 4)$$

The same construction is valid replacing k_3 by $k'_3 = -k_3$; the Postnikov tower $P' = ((\mathbb{Z}, 0), (\mathbb{Z}, k'_3))$ produces a *different* fibration:

$$K(\mathbb{Z}, 3) \hookrightarrow X' \twoheadrightarrow K(\mathbb{Z}, 2) \xrightarrow{-c_1^2} K(\mathbb{Z}, 4)$$

It is important to understand the fibrations not only are different but they are even non-isomorphic: their classifying maps are not *homotopic*. Yet the spaces $X = \mathbf{PS}(P)$ and $X' = \mathbf{SP}(P')$ are the same, that is, they have the same homotopy type; the following diagram is induced by the group morphism $\varepsilon_4 : K(\mathbb{Z}, 4) \xrightarrow{K(-1,4)} K(\mathbb{Z}, 4)$ associated to the symmetry $-1 : n \mapsto -n$ in $\mathbb{Z}$, and the same for ε_3.

$$\begin{tikzcd}
K(\mathbb{Z},3) \arrow[r] \arrow[d, "\varepsilon_3"', "\cong"] & X \arrow[r] \arrow[d, "\varepsilon_3\widetilde{\times}="', "\cong"] & K(\mathbb{Z},2) \arrow[r, "c_1^2"] \arrow[d, "="'] & K(\mathbb{Z},4) \arrow[d, "\varepsilon_4"', "\cong"] \\
K(\mathbb{Z},3) \arrow[r] & X' \arrow[r] & K(\mathbb{Z},2) \arrow[r, "-c_1^2"] & K(\mathbb{Z},4)
\end{tikzcd}$$

The $\cong$ sign between X and X' is particularly misleading. It is correct from the topological point of view: both spaces X and X' actually are homeomorphic and $\varepsilon_3\widetilde{\times}=$ is such a homeomorphism. The $\cong$ sign is incorrect with respect to the principal $K(\mathbb{Z}, 3)$-structures: the actions of $K(\mathbb{Z}, 3)$ on the fibres of X and X' are *not compatible*; the satisfied relation is only $(\varepsilon_3\widetilde{\times}=)(a \cdot x) = \varepsilon_3(a) \cdot (\varepsilon_3\widetilde{\times}=)(x)$ and the principal structures would be compatible if $(\varepsilon_3 \widetilde{\times}=)(a \cdot x) = a \cdot (\varepsilon_3 \widetilde{\times}=)(x)$ was satisfied, this is why the classifying maps are opposite.

Maybe the same phenomenon for the Hopf fibration is easier to be understood. Usually we take S^3 as the unit sphere of $\mathbb{C}^2$ so that a *canonical* S^1-action is underlying and a *canonical* characteristic class on the quotient S^3/S^1 is deduced. But if you reverse the S^1-action, why not, the space S^3 is not modified, the quotient S^3/S^1 is not modified either, but the characteristic class is the opposite one. In other words, it is important not to forget the classifying map characterizes the isomorphism class of a principal fibration, but not the homotopy type of the total space!

10.4.3 The key example.

The next example of a Postnikov tower with two stages is still rather simple but is sufficient to understand the essential failure of the claimed Postnikov *invariants*.

Let us consider the tower $P(\ell, k) = ((\mathbb{Z}^\ell, 0), (\mathbb{Z}, k))$, the parameter ℓ being some positive integer, and k, the unique non-trivial Postnikov "invariant" being an element $k \in H^4(K(\mathbb{Z}^\ell, 2), \mathbb{Z})$. A canonical isomorphism $K(\mathbb{Z}^\ell, 2) \cong K(\mathbb{Z}, 2)^\ell$ is available. The cohomoloy ring of $K(\mathbb{Z}, 2) = P^\infty\mathbb{C}$ is the polynomial ring $\mathbb{Z}[X]$ where $X = c_1$ is the first universal Chern class, of degree 2, so that $H^*(K(\mathbb{Z}^\ell, 2), \mathbb{Z}) = \mathbb{Z}[X_i]$ with $1 \leq i \leq \ell$, every generator X_i being of degree 2. Finally $H^4(K(\mathbb{Z}^\ell, 2), \mathbb{Z}) = \mathbb{Z}[X_i]^{[2]}$, the exponent [2] meaning we must consider only the sub-module of the homogeneous polynomials of degree 2 with respect to the X_i's. Every $k \in \mathbb{Z}[X_i]^{[2]}$ thus defines a two stages Postnikov tower $P(\ell, k) = ((\mathbb{Z}^\ell, 0), (\mathbb{Z}, k))$.

Two such *different* Postnikov towers $P(\ell, k)$ and $P(\ell', k')$ can be isomorphic. If so, the homotopy groups must me the same and $\ell = \ell'$ and it is enough to wonder whether $P(\ell, k) \overset{???}{\cong} P(\ell, k')$. A possible isomorphism $f : P(\ell, k) \to P(\ell, k')$ is made of $f_2 : \mathbb{Z}^\ell \xrightarrow{\cong} \mathbb{Z}^\ell$ and $f_3 : \mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$. The component f_3 is a possible simple sign change, as in the first example 10.4.2, but the component f_2 is a $\mathbb{Z}$-linear equivalence acting on the variables $[X_i]_{1 \leq i \leq \ell}$. The coherence condition given in Definition 156 becomes $f_{3*}(k) = {f_2}^*(k')$: the f_{3*} allows one to make equivalent two classes of opposite signs, and the ${f_2}^*$, much more interesting, allows one to make equivalent two classes $k, k' \in \mathbb{Z}[X_i]^{[2]}$ where k is obtained from k' by a $\mathbb{Z}$-linear change of variables. We have here identified f_2 with ϕ_2, the induced automorphism of $K(\mathbb{Z}^\ell, 2) = X_2$, the first stage of both Postnikov towers, see Definition 156.

Algebraic Topology succeeds: the topological problem of homotopy equivalence between $\mathbf{PS}(P(\ell, k))$ and $\mathbf{PS}(P(\ell, k'))$ is transformed into the algebraic problem of the $\mathbb{Z}$-linear equivalence, up to sign, between the "quadratic forms" k and k'. And this provides a complete solution, because this landmark problem firstly considered by Gauss has now a complete solution, see for example [59, 65, 14].

10.4.4 Higher dimensions.

But instead of working with the integer $3 = 2 * 2 - 1$, we could consider exactly the same problem with the Postnikov tower:

$$\mathcal{P}_{2d-1} \ni P(\ell, d, k) = ((\mathbb{Z}^\ell, 0), (0, 0), \ldots, (0, 0), (\mathbb{Z}, k_{2d-1} = k))$$

defined by integers $\ell \geq 1$, $d \geq 2$ and a cohomology class $k \in H^{2d}(K(\mathbb{Z}^\ell, 2), \mathbb{Z}) = \mathbb{Z}[X_i]^{[d]}$. Instead of an equivalence problem between homogeneous polynomials of degree 2, we meet the same problem but with homogeneous polynomials of degree d. And when this paper is written, this problem seems entirely *open* as soon as $d \geq 3$. Now is the right time to recall what the very notion of invariant is.

10.5 Invariants.

10.5.1 Elementary cases.

What is an *invariant*? An invariant is a process $\mathcal{I}$ which associates to every object X of some type some other object $\mathcal{I}(X)$, the relevant *invariant*; in other words, an invariant is a function. This terminology clearly says that $\mathcal{I}(X)$ *does not change* (does not *vary*) when X is replaced by X', if X and X' are equivalent in some sense: a possible relevant equivalence between X and X' should imply the *equality* – not again some other equivalence – between $\mathcal{I}(X)$ and $\mathcal{I}(X')$.

For example one of the most popular invariants is the set of *invariant factors* of square matrices. The concerned equivalence relation is the similarity. If K is a commutative field and $A \in M_n(K)$ is an $(n \times n)$-matrix with coefficients in K representing some endomorphism of K^n, the invariant factors of A are a sequence of polynomials $\phi(A) = (\mu_1, \ldots, \mu_k)$ characterizing *in this case* the similarity class of the matrix A: two matrices A and B are similar if and only if $\phi(A) \boxed{=} \phi(B)$. Another example is the minimal polynomial $\mu_1(A)$: if two matrices are similar, they have the *same* minimal polynomial. Idem for the characteristic polynomial which is the product of the invariant factors, and so on. It is well known that for example the characteristic polynomial does not characterize the similarity class, yet it is an invariant: if two matrices are similar, they have the *same* characteristic polynomial. Sometimes the characteristic polynomial is sufficient to disprove the similarity between two matrices, sometimes not. The trivial invariant consists in deciding that $\mathcal{I}(A) = *$, some fixed object, for every matrix; not very interesting but it is undoubtedly an… invariant. Symmetrically the tentative invariant $\mathcal{I}(A) = A$ is *not* an invariant, for there exist different (!) matrices[33].

Algebraic Topology is in a sense an enormous collection of (algebraic) invariants associated to topological spaces, invariants with respect to some equivalence

[33] See `http://encyclopedia.thefreedictionary.com/invariant` for other typical examples. Another amusing bug of the standard terminology in Algebraic Topology is the expression "characteristic class" in the classical fibration theory: the usual characteristic classes are actual invariants (!) of the isomorphism class but, except in simple situations, they do not characterize (!) this isomorphism class.

relation, frequently the homotopy equivalence. Typically a homotopy group π_n is an invariant of this sort. Not frequently, with respect to some appropriate equivalence relation, it is possible a *complete* invariant is available. For example the H_1 is a complete invariant for the homotopy type of a finite connected graph, the genus is a complete invariant for the diffeomorphism type of a closed orientable real manifold of dimension 2.

The last two examples, quite elementary, are interesting, because the difficult logical problem underlying this matter is often forgotten and easily illustrated in these cases. Let M_0 and M_1 be two closed orientable 2-manifolds that are diffeomorphic; if g denotes the genus, then $g(M_0) \boxed{=} g(M_1)$: the genus is an invariant; furthermore it is a complete invariant, because conversely $g(M_0) \boxed{=} g(M_1)$ implies both manifolds are diffeomorphic. We have framed the '=' sign, because the main problem in the continuation of the story is there.

Let us consider now the case of the finite graphs. In fact, it is *false* the H_1 functor is an invariant. If you take a triangle graph $G_0 = \triangle$ and a square graph $G_1 = \square$, same homotopy type, the careless topologist thinks $H_1(G_0) = H_1(G_1) = \mathbb{Z}$ so that H_1 looks like an invariant of the homotopy type, but it is important to understand this is deeply erroneous. With respect to any coherent formal definition of mathematics, in fact $H_1(\triangle) \neq H_1(\square)$, these H_1-groups are only *isomorphic.* To obtain an actual invariant of the homotopy type, you must consider the functor $\mathbf{H}_1 = \mathrm{IC} \circ H_1$, where IC is the "isomorphism class" functor, always difficult to properly define from a logical point of view, see for example [8]. But in the case of the H_1-group of a finite graph, it is a free $\mathbb{Z}$-module of finite type, it is particularly easy to determine whether two such groups are isomorphic and every topologist *implicitly* apply the IC functor without generating any error.

Such a situation is so frequent that most topologists come to confuse both notions of *functor* and *invariant*, and the case of the Postnikov "invariants" is rather amazing.

10.6 The alleged Postnikov "invariants".

10.6.1 Terminology.

We start with a sensible topological space, for example a finite simply connected CW-complex E. The textbooks explain how it is possible to define or sometimes to "compute" the Postnikov invariants $(k_n(E))_{n\geq 3}$. In our framework, the problem is the following:

Problem 157 — *How to determine a Postnikov tower* $P = ((\pi_n, k_n))_{n\geq 2}$ *such that* E *and* $\mathbf{PS}(P)$ *have the same homotopy type?*

This problem, thanks to the general *Constructive* Algebraic Topology framework of the authors, now has a positive *and* constructive solution. The aforementioned textbooks also describe "solutions", but which do *not* satisfy the constructive requirements which should yet be required in this context. See also [55] for another theoretical *constructive* – and interesting – solution, significantly more complex, so that it has not yet led to concrete results, that is, to machine programs.

Most topologists think a positive solution for Problem 157 imply the k_n's of the result are "invariants" of the homotopy type of E. This is simply *false*, for any reasonable understanding of the word *invariant*, and it is rather strange such an error remains present a so long time in a so important field as basic Algebraic Topology. The k_n's could be called *invariants* if they solved the next problem.

Problem 158 — Construct a functor $\mathbf{SP} : \mathcal{SS}_{EH} \to \mathcal{P}$ satisfying the following properties:

1. Some original space $E \in \mathcal{SS}_{EH}$ and $\mathbf{PS} \circ \mathbf{SP}(E)$ have the same homotopy type;
2. If E and $E' \in \mathcal{SS}_{EH}$ have the same homotopy type, then $\mathbf{SP}(E) \boxed{=} \mathbf{SP}(E')$.

The first point is a rephrasing of Problem 157, and the second states that if E and E' have the same homotopy type, then the images $\mathbf{SP}(E)$ and $\mathbf{SP}(E')$ are $\boxed{\text{equal}}$, not only *isomorphic*. In other words the claimed "invariant" must not *change* when the source object changes in the same equivalence class; this is of course (?) the very notion of invariant.

The non-constructive topologist easily solves the problem by replacing the category $\mathcal{P}$ by the quotient $\mathcal{P}/\text{Iso}$, and then a correct solution is obtained, but it is an artificial one. The category $\mathcal{SS}_{EH}/H$-equiv and the canonical projection $\mathcal{SS}_{EH} \to \mathcal{SS}_{EH}/H$-equiv would be much simpler, but obviously without any interest.

The right interpretation of the k_n's is the following: combined with the standard homotopy groups π_n, they are to be considered as *directions for use* allowing one to reconstruct a simple object with the right homotopy type; another rephrasing of Problem 157. But it can happen two different objects E and E' with the *same* homotopy type produce *different* "directions for use", so that these "directions for use" are not invariants of the homotopy type. In fact such an accident is the most common situation, except for the topologists working only with paper and pencil.

10.6.2 The SP functor, first try.

Let us briefly describe the standard solution of Problem 157, a solution which can be easily made constructive thanks to [56, 52, 55]. Let E be some reasonable[34] simply connected space. There are many ways to determine the[35] Postnikov tower $P = \mathbf{SP}(E)$ and one of them is illustrated here with the beginning of the simplest case, the 2-sphere S^2. Hurewicz indicates $\pi_2 = H_2 = \mathbb{Z}$; the invariant k_2 is necessarily null. The next step invokes the Whitehead fibration:

$$K(\mathbb{Z}, 1) \hookrightarrow E^3 \twoheadrightarrow S^2 \xrightarrow{c_1} K(\mathbb{Z}, 2).$$

[34]That is, an $\mathcal{SS}_{EH}$-object, see [52].

[35]In fact *some* Postnikov tower…

where c_1 is the canonical cohomology class, in this case the first Chern class of the complex structure of S^2. The first stage of the Postnikov tower is $X_2 = K(\mathbb{Z}, 2) = P^\infty\mathbb{C}$ and the first stage of the complementary Whitehead tower is the total space $E^3 = S^3$: our fibration is nothing but the Hopf fibration. Then $\pi_3(S^2) = \pi_3(S^3) = H_3(S^3, \mathbb{Z}) = \mathbb{Z}$, so that the next Postnikov invariant is some $k_3 \in H^4(X_2, \mathbb{Z}) = H^4(K(\mathbb{Z}, 2), \mathbb{Z}) = \mathbb{Z}$. How to determine this cohomology class?

In general we obtain a fibration:

$$E^n \hookrightarrow E \to X_{n-1}$$

where X_{n-1} is the $(n-1)$-stage of the Postnikov tower containing the homotopy groups $(\pi_i)_{2\leq i\leq n-1}$, and E^n is the complementary n-stage of the Whitehead tower [18, Proposition 8.2.5] containing the homotopy groups $(\pi_i)_{i\geq n}$; in the Kan context of [40, § 8], E^n is the n-th Eilenberg subcomplex of E. How to deduce a cohomology class $k_n \in H^{n+1}(X_{n-1}, \pi_n)$? The $(n-1)$-connectivity of E^n produces a transgression morphism $H^n(E^n, \pi_n) \to H^{n+1}(X_{n-1}, \pi_n)$; the group $H^n(E^n, \pi_n)$ contains a fundamental Hurewicz class and the image of this class in $H^{n+1}(X_{n-1}, \pi_n)$ is the wished k_n. In the particular case of S^2 this process leads to an isomorphism $H^3(S^3, \mathbb{Z}) \xrightarrow{\cong} H^4(K(\mathbb{Z}, 2), \mathbb{Z})$ so that k_3 is the image of the fundamental cohomology class of S^3, that is, the (?) generator c_1^2 of $H^4(K(\mathbb{Z}, 2), \mathbb{Z})$. Sure?

As usual we have light-heartedly mixed intrinsic objects and isomorphism classes of these objects. The *correct* isomorphism to be considered for our example is $H^3(E^3, \pi_3(\underline{E^3})) \cong H^4(K(\pi_2(\underline{S^2}), 2), \pi_3(\underline{E^3}))$ where E^3 is now the total space of the *canonical* fibration $K(\pi_2(\underline{S^2}), 1) \hookrightarrow E^3 \twoheadrightarrow S^2$; this isomorphism actually is canonical. But no canonical ring structure for $\pi_3(E^3)$ so that speaking of c_1^2 does not make sense. There is actually a canonical element $k_3 \in H^4(K(\pi_2(\underline{S^2}), 2), \pi_3(\underline{E^3}))$, but such an element deeply depends on S^2 itself and cannot be qualified as an *invariant* of the *homotopy type* of S^2. An actual invariant should be taken in the "absolute" (independent of S^2) group $H^4(K(\mathbb{Z}, 2), \mathbb{Z})$, but such a choice depends on an isomorphism $\pi_3(E^3) \cong \mathbb{Z}$; two such isomorphisms are possible so that in this case the k_3 is defined *up to sign*: it is well known the Hopf fibration and the "opposite" one produce the "same" total space.

This is the reason why in the definition of a Postnikov tower, see Definition 152, we have decided to have only *one* group for each isomorphism class; this is easy and can be done in a *constructive* way. The goal being to obtain *invariants*, we had to design our Postnikov towers as a catalogue of possible Postnikov towers, in such a way that there are no redundant copies up to isomorphism in this collection; bearing this point in mind, it was mandatory to have only one copy for every isomorphism class of group. But this was not enough, for it is today impossible to take the same precaution for the second components, the k_n's, the so-called Postnikov invariants.

For example if the concerned homotopy groups are finite, then the number of possible k-invariants is finite, so that the related equivalence problem is theoretically solved; this was already noted by Edgar Brown [9], which conversely implies (!) he did not know how to solve the general case. On the contrary, as soon as the homotopy groups have infinite automorphism groups, there is no known way to transform the pseudo-invariants into actual invariants.

We understand now the reason of the repetitive remark in Section 10.3: “In this particular case, the k_n actually is an invariant of the homotopy type”; we decided to systematically choose $\pi_n = \mathbb{Z}_2$, but the automorphism group of $\mathbb{Z}_2$ is *trivial*; no non-trivial automorphism of the constructed tower can exist and *then* the k_n's are actual *invariants*.

But if some user intends to use the Postnikov invariants to try to prove the spaces E and E' have different homotopy types, the following accident can happen. A calculation could respectively produce the Postnikov towers $((\mathbb{Z}^\ell, 0), (0, 0), \ldots, (\mathbb{Z}, k_{2d-1}))$ and $((\mathbb{Z}^\ell, 0), \ldots, (\mathbb{Z}, k'_{2d-1}))$ (see Section 10.4.4). If fortunately $k_{2d-1} = k'_{2d-1}$ our user can be sure the homotopy types are the same but if on the contrary $k_{2d-1} \neq k'_{2d-1}$, then he has to decide whether two homogeneous polynomials of degree d are linearly equivalent or not and for $d \geq 3$: no general solution is known. *Maybe* they are equivalent, *maybe* not; because the alleged invariants may… vary, in general our user cannot conclude: the claimed invariants cannot play the role ordinarily expected for invariants; qualifying them as invariants is therefore a deep *error*.

10.6.3 The SP functor, second try.

The right definition for the **SP** functor is now clear. We must add to the data some *explicit* isomorphisms between the homotopy groups $\pi_n(E)$ of the considered space E with the corresponding *canonical* groups, see Definition 152.

Definition 159 — The product $\mathcal{SS}_{EH} \widetilde{\times} I$ is the set of pairs (E, α) where:

1. The E component is a simplicial set with effective homology $E \in \mathcal{SS}_{EH}$;
2. The α component is a collection $(\alpha_n)_{n \geq 2}$ of isomorphisms $\alpha_n : \pi_n(E) \xrightarrow{\cong} \pi_n$ where π_n denotes the unique *canonical* group isomorphic to $\pi_n(E)$.

The previous discussions of this text can reasonably be considered as a demonstration of the next theorem.

Theorem 160 — *A functor* $\mathbf{SP} : \mathcal{SS}_{EH} \widetilde{\times} I \to \mathcal{P}$ *can be defined.*

1. *If* $(E, \alpha) \in \mathcal{SS}_{EH} \widetilde{\times} I$*, then* E *and* $\mathbf{PS} \circ \mathbf{SP}(E, \alpha)$ *have the same homotopy type.*
2. *If* $P \in \mathcal{P}$ *is a Postnikov tower, there exists a unique* α *such that* $\mathbf{SP}(\mathbf{PS}(P), \alpha) = P$.

So that it is tempting – and correct – to replace the **PS** functor by another one $\mathbf{PS} : \mathcal{P} \to \mathcal{SS}_{EH} \widetilde{\times} I$ to obtain a better symmetry. But the ordinary topologists work with elements in $\mathcal{SS}_{EH}$, not in $\mathcal{SS}_{EH} \widetilde{\times} I$.

10.7 The Postnikov invariants in the available literature.

Most textbooks speaking of Postnikov invariants (or k-invariants) use the *invariant* terminology without justifying it, so that strictly speaking, no mathematical error in this case. For example [18, p. 279] defines the Postnikov invariant through a transgression morphism[36] and explains "The k^i precisely constitute the stepwise obstructions…"; the statement about this obstruction of course is correct but it seems the terminology should therefore speak of Postnikov *obstructions*? Nothing is explained about the *invariant* nature of these obstructions.

Other books speak of these invariants as objects allowing to *reconstruct* the right homotopy type. For example, in [29, p. 412]: "The map k_n is equivalent to a class in $H^{n+2}(X_n; \pi_{n+1}(K))$ called the n-th *k-invariant* of X. These classes specify how to construct X inductively from Eilenberg-MacLane spaces". To be compared with our considerations about the interpretation in terms of "directions for use" at the end of Section 10.6.1. Again, no indication in this book about the justification of the *invariant* terminology. The Section "The Postnikov Invariants" of [17, V.3.B] can be analyzed along the same lines.

In [24, VI], because of a sophisticated categorical environment, the authors prefer to define the *general* notion of *Postnikov tower* for a space X, each one containing in particular its k_n-invariants [24, VI.5]; finally Theorem [24, VI.5.14] proves two such Postnikov towers for the same X are *weakly* equivalent. In other words one source object produces in general a large infinite set of (different!) k_n-invariants, for every relevant n; yet some invariant theory is interesting when different objects can produce the same invariants, not when an object produces different invariants! In fact, as explained in our text, this cannot be currently avoided, but why these authors do not make explicit the misleading status of these claimed invariants?

The book [40] systematically uses the powerful notion due to Kan of *minimal simplicial Kan-model*, often allowing a user to work in a "canonical" way, allowing frequently the same user to easily detect a non-unicity problem. In this way [40, p. 113] correctly signals that the map $B \to K(\pi, n+1)$ leading to a k_n-invariant is defined up to a π-automorphism, which is not a serious drawback: the decision problem about the possible equivalence of two k_n's under such an automorphism is easy when π is of finite type. But the author does not mention the same problem with respect to the automorphisms of the base space B, the automorphisms leading to the corresponding open problem detailed here Section 10.4.4.

The same author in a more recent textbook [41] again considers the same question. He defines the notion of Postnikov system in Section 22.4; the existence of *some* Postnikov system is proved, the terminology k-invariant is used one time, between quotes seeming imply this expression is not really appropriate, but without any explanations.

Hans Baues [6, p.33] on the contrary correctly respects the necessary symmetry between the source and the target of the classifying map; but the author is aware

[36] We used this method in Section 10.6.2

of the underlying difficulty and it is interesting to observe how he "solves" the raised problem:

> Here $k_n(Y)$ is actually an *invariant* of the homotopy type of Y in the sense that a map $f : Y \to Z$ satisfies:
>
> $$(P_{n-1}f)^* k_n(Z) = (\pi_n f)_* k_n(Y)$$
>
> in $H^{n+1}(P_{n-1}X, \pi_n Y)$.

Clearly explained, the author says that the *invariant* is variable, but in a *functorial* way. Baues' condition is essentially the *coherence condition* of our Definition 156. If the appropriate morphisms of the category $\mathcal{SS}_{EH}\widetilde{\times}I$ were defined, the functorial property of the map **SP** (Theorem 160) would be exactly Baues' relation. But it is not explained in Baues' paper why a functor may be qualified as an invariant.

Probably the reference the most lucid about our subject is [66]. Chapter IX is entirely devoted to Postnikov systems. We find p. 423:

> The term 'invariant' is used somewhat loosely here. In fact k^{n+2} is a cohomology class of a space X^n, which has not been constructed in an invariant way. This difficulty, however, is not serious, for, as we shall show below, the construction of the space X^n can be made completely natural.

This text is essentially a rephrasing of Baues' explanation. Again the common confusion between the notions of invariant and functor is observed. To make "natural" its invariants, George Whitehead uses enormous singular models, so that the obtained k^{n+2} heavily depends on X itself and not only on its homotopy type. In fact Section [66, IX.4] shows Whitehead is in fact also interested in being able to reconstruct the homotopy type of X from the "natural" associated Postnikov tower, and this goal is obviously reached, but this does not provide a general machinery allowing one to detect *different homotopy types* when the associated *invariants* are *different*.

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